

The University of Chicago

COMPOSITION AND RANK OF n -WAY
MATRICES AND MULTI-
LINEAR FORMS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1934

By
RUFUS OLDENBURGER

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COMPOSITION AND RANK OF n -WAY MATRICES AND MULTILINEAR FORMS¹

BY RUFUS OLDENBURGER

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1. **Introduction.** Zehfuss² proved that a full sign determinant of even class of a system of forms is a relative invariant of these forms, and showed that the full sign determinant of an n -adic form F of even degree is an invariant of index equal to the degree n of F . Hitchcock³ proved that the 2-way ranks of polyadics are invariant under non-singular linear transformations. The matrices associated with polyadics possess certain factorization properties,⁴ i.e., if $A = (a_{i_1 \dots i_p})$ is the matrix of a polyadic, then

$$A = \left(\sum_{j=1}^h a_{1ji_1} \dots a_{pji_p} \right),$$

where $(a_{1ji_1}), \dots, (a_{pji_p})$ are 2-way matrices. Rice⁵ proved certain bound properties of 2-way ranks of general n -way matrices. Hitchcock⁶ further proved by polyadic methods the invariance of certain space ranks of the matrix A above under non-singular linear transformations.

The present paper begins with the study of certain operations on n -way matrices which the author has designated *compositions*. These operations include sets of linear transformations on one or more indices, Cremona transformations, the transformations of the characteristic matrices of a linear algebra, the transformations of tensors and polyadics, etc. Bound properties of 2-way ranks under compositions, relations between the ranks of a given matrix, and the canonical forms which depend on these ranks are established. Space ranks of a general signancy are introduced and several properties of these ranks are obtained. Finally, the invariance of ranks of compound matrices under non-singular compositions is proven by the use of multiple concomitants.

By means of the equivalence of a set of transformations to a single composition,

¹ Presented to the American Mathematical Society, December, 1930.

² Zehfuss, Ueber die Erweiterung des Begriffes der Determinanten, Programme Gewerbeschule Frankfurt a/M. (1868), p. 6.

³ F. L. Hitchcock, The expression of a tensor or a polyadic as a sum of products, Journal of Mathematics and Physics, vol. 6 (1927), pp. 164-189. We shall call this paper "Hitchcock, Paper 1."

⁴ F. L. Hitchcock, Ordered determinants with application to polyadics, Journal of Mathematics and Physics, vol. 4 (1925), pp. 222, 223.

F. L. Hitchcock, Paper 1, p. 164.

⁵ L. H. Rice, Couche ranks in the general matrix, Journal of Mathematics and Physics, vol. 7 (1928), pp. 94-96.

⁶ F. L. Hitchcock, Multiple invariants and generalized rank of a p -way matrix or tensor, Journal of Mathematics and Physics, vol. 7 (1927), pp. 40-79.

the author is able to show how the complete invariant theory of 2-way matrices can be extended to n -way matrices.

Throughout the paper repetition of subscripts will denote summation.

Unless otherwise stated the proofs hold for an arbitrary field of numbers commutative under multiplication.

2. Definitions. It is assumed that the reader is familiar with the terminology used in standard works on ordinary (2-way) matrices. We shall also need the following terms.

A *display* $(a_{i_1 i_2 \dots i_p})$ of a p -way matrix A is defined to be the array of numbers obtained by placing $a_{i_1 i_2 \dots i_p}$ at the point $(i_1, i_2, \dots, i_p)$ in a linear p -space where $i_j = 1, 2, \dots, n_j$ for $j = 1, 2, \dots, p$. The j -th direction in this space is the direction associated with the index i_j . Other p -way displays of A are obtained by transpositions of the indices in $(a_{i_1 i_2 \dots i_p})$.

A set of indices of A determines an *aspect*⁷ or orientation in the space of A , a single index determining a direction in this space. An ordered set of indices $T = i_r i_s \dots i_u i_w$ of the indices of A is called the *partition* T . A partition is of *class* n if it contains n indices. The class of the partition containing all of the indices of A is called the class of A . The *order* of a partition T is the product of the orders of the ranges of the indices contained in T . A partition σ will be called a *subset* of T if the indices of σ are contained in T . Two partitions T_1, T_2 will be said to be equal if they have the same ranges on corresponding indices (1st, 2nd, $\dots$). If ρ is the partition $i_r i_s \dots$, σ is the partition $i_q i_t \dots$, $\dots$, γ is the partition $i_u i_w \dots$, where $T = i_r i_s \dots i_q i_t \dots i_u i_w \dots$, then $\rho, \sigma, \dots, \gamma$ are said to simply cover the partition T . If $\rho, \sigma, \dots, \gamma$ simply cover T , we write $T = \rho \sigma \dots \gamma$.

Let $T_1, T_2, \dots, T_r$ be mutually exclusive, exhaustive partitions of the indices of A . We define the display $(a_{T_1 T_2 \dots T_r})$ of A to be the matrix obtained by letting $T_1, \dots, T_r$ denote the indices of an r -way matrix, indices in each partition varying by convention from right to left. For example, if $A = (a_{i_1 i_2 i_3 i_4})$, $i_1, i_2, i_3, i_4 = 1, 2$, and $T_1 = i_1 i_2, T_2 = i_3 i_4$, then

$$A = (a_{T_1 T_2}) = \begin{bmatrix} a_{1111} & a_{1112} & a_{1121} & a_{1122} \\ a_{1211} & a_{1212} & a_{1221} & a_{1222} \\ a_{2111} & a_{2112} & a_{2121} & a_{2122} \\ a_{2211} & a_{2212} & a_{2221} & a_{2222} \end{bmatrix}.$$

In the general case where $r = 2$, we obtain the display $(a_{T_1 T_2})$ which possesses a 2-way *determinant* $|a_{T_1 T_2}|$ on T_1, T_2 provided that the orders of T_1 and T_2 are equal. A matrix is non-singular on T_1, T_2 if $|a_{T_1 T_2}| \neq 0$.

⁷ This term was introduced by L. H. Rice in his paper entitled Couche ranks in the general matrix, Journal of Mathematics and Physics, vol. 7 (1928), pp. 94 ff.

An asterisk on T will indicate that the indices of T have been assigned fixed values (T^*). If every index of T is given the value 1 we write $T = 1$ (omitting the asterisk); similarly we write $T = 2$, etc. A *minor* of A is a matrix obtained from A by allowing the indices of A to vary over subsets of their ranges. A j -way layer of A is a minor of A obtained by fixing $(n - j)$ indices, where n is the class of A . A layer obtained by letting $T = T^*$ is called a T -layer.

A *cube* matrix is a matrix which has the same range on each index.

The usual notions concerning zero matrices, equality of matrices, etc. will apply here.

3. Compositions. Let two matrices A and B be displayed as $(a_{T_1 T})$, $(b_{T T_2})$ respectively. The matrix $C = (a_{T_1 T} b_{T T_2})$ is the *composite* of A and B on T , and is denoted by $A | T | B$. C is said to be obtained from A and B by composition on T . If T contains no indices, the composition is a *zero-composition*, and following the notation used by Weyl⁸ in treating the two way case it is denoted by $A \times B$.

By a method analogous to that used by Hilbert,⁹ the form

$$(1) \quad F = a_{i_1 \dots i_n T} b_{T i_{n+1} \dots i_{n+m}} x_{i_1}^1 \dots x_{i_{n+m}}^{n+m}, \quad T = j_1 \dots j_l,$$

can be obtained from the forms

$$G = a_{i_1 \dots i_n i_1 \dots i_l} x_{i_1}^1 \dots x_{i_n}^n y_{j_1}^1 \dots y_{j_l}^l,$$

$$H = b_{j_1 \dots j_l i_{n+1} \dots i_{n+m}} y_{j_1}^{l+1} \dots y_{j_l}^{2l} x_{i_{n+1}}^{n+1} \dots x_{i_{n+m}}^{n+m},$$

by applying the following differential operator

$$\sum_{y_{j_1}^1 \dots y_{j_l}^{2l}} \frac{\partial^{2l}}{\partial y_{j_1}^1 \dots \partial y_{j_l}^l \partial y_{j_1}^{l+1} \dots \partial y_{j_l}^{2l}}$$

to GH . Several properties of compositions follow at once from the properties of differential operators. Now if the coefficients are sufficiently restricted F can be obtained by making the homogeneous transformation¹⁰

$$y_{j_1}^1 \dots y_{j_l}^l = b_{j_1 \dots j_l i_{n+1} \dots i_{n+m}} x_{i_{n+1}}^{n+1} \dots x_{i_{n+m}}^{n+m}$$

on the variables of G .

If $A = (a_{i_1 \dots i_n})$, $B = (b_{i_1 \dots i_n})$, we define $A \pm B$ to be the matrix $(a_{i_1 \dots i_n} \pm b_{i_1 \dots i_n})$. If T_1, T_2 are disjoint, $A | T_1 | (B | T_2 | C) = (A | T_1 | B) | T_2 | C$. Evidently $A | T_1 | (B \pm C) = A | T_1 | B \pm A | T_1 | C$.

⁸ H. Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, 1931, pp. 80 ff.

⁹ D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig, 1912, pp. 128, 129.

¹⁰ Cremona transformations are contained in the class of homogeneous transformations.

4. Identity matrices. If in the matrix $\delta = (\delta_{T_1 T_2})$, $T_1 = T_2$, $\delta_{T_1 T_2}^* = 1$ if $T_1^* = T_2^*$, and $\delta_{T_1 T_2}^* = 0$ if $T_1^* \neq T_2^*$, then δ is said to be an identity matrix on T_1, T_2 . Evidently if $(a_{T_2 T_3}) = (a_{T_1 T_3})$, $(\delta_{T_1 T_2}) \mid T_2 \mid (a_{T_2 T_3}) = (a_{T_1 T_3})$.

If t is a number, $t \times \delta$ is a scalar matrix.

5. 2-way ranks. In a matrix A the T -layers $A_1, A_2, \dots, A_r$ are said to be linearly independent if and only if

$$c_f \times A_f = 0$$

implies that $c_f = 0$ for $f = 1, 2, \dots, r$. The T -rank of A is the maximum number of linearly independent T -layers of A . Evidently the T^s -rank of A is the same as the T -rank of A where T^s is obtained from T by making a substitution S on the indices of T . Further the T_1 -rank of A is the rank of the 2-way display $(a_{T_1 T_2})$, and is hence called a 2-way rank (assuming that T_2 is not vacuous). If T_2 is vacuous, the T_1 -rank of $A = (a_{T_1})$ is 1, or zero.

A matrix A is said to be *singular* on T if the T -rank of A is less than the order of T . If α and T are disjointed, the α -rank of A is said to be *invariant* under T -composition with C if the α -rank of A is equal to the α -rank of $A \mid T \mid C$.

6. Properties of 2-way ranks under general compositions. The reduction of the problem of properties of a set of compositions to the problem of properties of a single composition is accomplished by means of the lemma below.

Let matrices A_i possess determinants $|A_i|$ on T_{i1}, T_{i2} for $i = 1, 2, \dots, n$. Let D represent the determinant $|A_1 \times A_2 \times \dots \times A_n|$ whose rows are denoted by the partition $T_{11} \dots T_{n1}$ and columns by $T_{12} \dots T_{n2}$. Finally let Π_i denote the order of T_{i1} . With these notations we have the

LEMMA. The determinant $D = |A_1 \times A_2 \times \dots \times A_n| = |A_1|^{\Pi_2 \dots \Pi_n} |A_2|^{\Pi_1 \Pi_3 \dots \Pi_n} \dots |A_n|^{\Pi_1 \dots \Pi_{n-1}}$.

Weyl¹¹ has shown that

$$(A_1 \times \delta_1) \mid T_{12} T_{21} \mid (\delta_2 \times A_2) = A_1 \times A_2,$$

where δ_1, δ_2 are identity matrices on partitions T_{21}, T'_{21} and T_{12}, T'_{12} respectively. The matrix $A_1 \times \delta_1$ displayed with T_{11}, T'_{21} as the partition of its rows is a diagonal matrix with Π_2 minors equal to A_1 on the diagonal. Hence $|A_1 \times \delta_1| = |A_1|^{\Pi_2}$. Also $|A_2 \times \delta_2| = |A_2|^{\Pi_1}$, whence $|A_1 \times A_2| = |A_1|^{\Pi_2} |A_2|^{\Pi_1}$. The Lemma follows by induction.

Since the matrices $B_i = (b_{T_{i1} T_{i2}})$ of composition on $T_{i1} = 1, 2, \dots, n$ form a matrix $(e_{T_{11} T_{21}} \dots T_{n1}, T_{12} T_{22} \dots T_{n2}) = (b_{T_{11} T_{12}}) \times (b_{T_{21} T_{22}}) \times \dots$ of composition on $T = T_{11} \dots T_{n1}$, we have the

COROLLARY. If a matrix $A = (a_{T_1 T_2})$ possesses a determinant $|a_{T_1 T_2}|$, the

¹¹ H. Weyl, Ibid.

corresponding determinant of C obtained from A under composition on T_{i1} ($i = 1, \dots, n$) simply covering T_2 is equal to

$$|a_{T_1 T_2}| |b_{T_{11} T_{12}}|^{\Pi_2 \cdots \Pi_n} \cdots |b_{T_{n1} T_{n2}}|^{\Pi_1 \cdots \Pi_{n-1}},$$

where $B_i = (b_{T_{i1} T_{i2}})$ are the matrices of composition. The B_i are assumed to be square as displayed.

By letting $B_{n-j+1}, \dots, B_n$ be identity matrices on $T_{n-j+1,1}, T_{n-j+1,2}, \dots, T_{n1}, T_{n2}$, it is seen that compositions on $T_{11}, T_{21}, \dots, T_{n-j,1}$ forming a subset of T_2 are also treated by the corollary.

The following theorem is obvious.

THEOREM 1. The T and σ -ranks of $A = (a_{T\sigma})$ are equal.

We also have

THEOREM 2. If T_1 and T_3 are disjointed and T_2 is a subset of T_3 , the T_1 -rank of a T_3 -layer of A is less than or equal to the T_1 -rank of a T_2 -layer of A .

The following theorem is a slight generalization of a theorem of Rice.¹²

THEOREM 3. If $T = T_1 \cdots T_s$, the T -rank of A is less than or equal to the product of the T_i -ranks of A where $i = 1, \dots, s$.

By elementary compositions on A we shall mean elementary transformations¹³ on the layers of A . By these compositions, if r_i is the T_i -rank of A , r_i of the T_i -layers of A can be reduced to zero layers, leaving by Theorem 6 the $T, T_1, \dots, T_s$ -ranks of A invariant. A is then reduced to a matrix in which there are at most $r_1 \cdots r_s$ non-zero intersections of the $T_1, \dots, T_s$ -layers. But these intersections are T -layers, whence the theorem is proved.

We have

THEOREM 4. Let three matrices A, B, C be given by $(a_{T_1 T_2 T_3}), (b_{T_3 T_4 T_5}), A | T_3 | B$ respectively. Let r'_i, r''_i , and r_i represent the T_i -ranks of A, B , and C respectively. Let further r_{25} be the $T_2 T_5$ -rank of C . Then $r_{25} \leq r'_2 r''_5$.

By theorem 3 $r_{25} \leq r_2 r_5$. Now $C = A | T_1 T_3 | (\delta \times B)$, where $\delta = (\delta_{T_1 T'_1})$ is an identity matrix on T_1, T'_1 . Since C is the ordinary product of two 2-way matrices, $r_2 \leq r'_2$. Similarly $r_5 \leq r''_5$.

THEOREM 5. Let A and B be defined as in Theorem 4. If B is non-singular on T_3 , the T_2 -rank of A is invariant under composition on T_3 with B .

If $\delta = (\delta_{T_1 T'_1})$, as above, by the Lemma $\delta \times B$ is non-singular on $T_1 T_3$. Hence $r_2 = r'_2$.

THEOREM 6. The $T_i T_j \cdots T_l$ -rank $r_{ij \dots l}$ of $A = (a_{T_1 \dots T_s})$ is invariant under a set of compositions on $T_1, \dots, T_s$ with matrices $C_1 = (c_{T_1 \sigma_1}), \dots, C_s = (c_{T_s \sigma_s})$, where C_i is non-singular on T_i for $i = 1, 2, \dots, s$.

Let $E = (e_{\alpha, \beta})$ be defined to be the matrix

$$(e_{T_i T_j \dots T_l, \sigma_i \sigma_j \dots \sigma_l}) = (c_{T_i \sigma_i} c_{T_j \sigma_j} \cdots c_{T_l \sigma_l}),$$

and

$$F = (f_{T-\alpha, T-\beta}) = (c_{T_1 \sigma_1} c_{T_2 \sigma_2} \cdots c_{T_{i-1} \sigma_{i-1}} c_{T_{i+1} \sigma_{i+1}} \cdots c_{T_{j-1} \sigma_{j-1}} c_{T_{j+1} \sigma_{j+1}} \cdots),$$

¹² L. H. Rice, Ibid.

¹³ See Bocher's Introduction to Higher Algebra, p. 55.

where $T - \alpha$, α simply cover $T = T_1 \cdots T_s$, and $\sigma - \beta$, β simply cover $\sigma = \sigma_1 \cdots \sigma_s$. Now F is non-singular on $T - \alpha$ for we can choose subsets $\sigma'_1, \sigma'_2, \dots$, of $\sigma_1, \sigma_2, \dots$ respectively such that $(c_{T_1\sigma'_1})$ is non-singular on T_1, σ'_1 , $(c_{T_2\sigma'_2})$ is non-singular on $T_2, \sigma'_2, \dots$. By the Lemma the minor $(c_{T_1\sigma'_1} c_{T_2\sigma'_2} \cdots c_{T_{i-1}\sigma'_{i-1}} c_{T_{i+1}\sigma'_{i+1}} \cdots)$ of F is non-singular on $T - \alpha, \gamma$ where $\gamma = \sigma'_1 \sigma'_2 \cdots \sigma'_{i-1} \sigma'_{i+1} \cdots$. By Theorem 5 $r_{ij \dots l}$ is invariant under composition with F on $T - \alpha$. Similarly E is non-singular on α , whence we again apply Theorem 5.

The following Corollary to Theorem 6 was proved by Hitchcock¹⁴ for polyadics.

COROLLARY. *The 2-way ranks of a matrix A are invariant under non-singular linear transformations on the indices of A .*

7. Properties of ranks under zero-composition. We are able to reduce the problem of ranks under zero-compositions to the problem under T -compositions where T is not vacuous. Let $C = A \times B$, where $C = (c_{\alpha\beta})$, $A = (a_\alpha)$, and $B = (b_\beta)$. Evidently $C = A' | i | B'$, where $A' = (a'_{\alpha 1})$, and $B' = (b'_{\beta i})$; furthermore i ranges over the single value 1, $a'_{\alpha 1} = a_\alpha$, and $b'_{\beta 1} = b_\beta$. The ranks of A and B are equal to the corresponding ranks of A' and B' .

Keeping the conventions of Theorem 4 concerning the r 's we have

THEOREM 7. *If $A = (a_{T_1 T_2})$, $B = (b_{T_3 T_4})$, the $T_2 T_4$ -rank r_{24} of $A \times B$ satisfies the condition*

$$r_{24} \leq r'_2 r''_4.$$

The Theorem follows from Theorem 4 and the representations which involve C above.

COROLLARY 1. *If $A = (a_\alpha)$, $B = (b_\beta)$, the α - and β -ranks of the composite $A \times B$ are both unity or zero.*

For polyadics Corollary 1 is a consequence of a theorem of Hitchcock.¹⁵

By Theorem 4, $r_2 \leq r'_2$ where r_2, r'_2 are the T_2 -ranks of $A \times B, A$ respectively. If B is not zero, some element $b_{T_3 T_4} \neq 0$. Hence the T_2 -rank of the layer $(a_{T_1 T_2} b_{T_3 T_4})$ is r'_2 . We therefore have

COROLLARY 2. *If $A = (a_{T_1 T_2})$, $B = (b_{T_3 T_4})$ are not zero, the T_2 -rank of $A \times B$ is equal to the T_2 -rank of A .*

THEOREM 8. *If $A = (a_T)$ is not zero, and $B = (b_{\rho\sigma})$, the $T\sigma$ -rank of $A \times B$ is equal to the σ -rank of B .*

Under composition on T with a matrix $C = (c_{T\alpha})$ non-singular on T, α the matrix $A \times B$ can be reduced to $A' \times B$ where A' contains an element unity while all other elements vanish. The $\alpha\sigma$ -rank of $A' \times B$ is equal to the σ -rank of B . By Theorem 5 the $\alpha\sigma$ -rank of $A' \times B$ is equal to the $T\sigma$ -rank of $A \times B$, whence Theorem 8 follows.

¹⁴ F. L. Hitchcock, The Expression of a Tensor or Polyadic as a sum of Products, Journal of Mathematics and Physics, vol. 6 (1927), p. 177.

¹⁵ Hitchcock, Ibid.

8. **Inverse matrices.** Let $A = (a_{T_1 T_2})$ be non-singular on T_1, T_2 . The inverse of A on T_1, T_2 is defined to be the matrix $A_{T_1 T_2}^{-1}$ which satisfies the relations

$$(2) \quad A | T_1 | A_{T_1 T_2}^{-1} = \delta_{T_2}; \quad A | T_2 | A_{T_1 T_2}^{-1} = \delta_{T_1},$$

where $\delta_{T_2} = (\delta_{T_2 T'_2})$ is an identity matrix on T_2, T'_2 , and $\delta_{T_1} = (\delta_{T_1 T'_1})$ is an identity matrix on T_1, T'_1 .

THEOREM 9. *If a matrix $A = (a_{T_1 T_2})$ is non-singular on T_1, T_2 , there exists a unique inverse matrix*

$$A_{T_1 T_2}^{-1} = (b_{T_2 T_1}),$$

where $b_{T_2 T_1}$ is the quotient of the cofactor of the element $a_{T_1 T_2}$ in the 2-way display $(a_{T_1 T_2})$ by the determinant of A on T_1, T_2 .

9. **Matrices with ranks equal to unity.** The following theorem has been proved by Hitchcock¹⁶ for polyadics.

THEOREM 10. *If the T -rank of a matrix $A = (a_{T\sigma})$ is unity, then A is the zero composite of two matrices one of whose partitions is the partition T .*

Evidently there exist matrices $B = (b_{\alpha T})$, $C = (c_{\sigma\beta})$, non-singular on α, T and σ, β respectively, such that

$$(3) \quad B | T | A | \sigma | C = A',$$

where $A' = (a'_{\alpha\beta})$, $a'_{\alpha^* \beta^*} = 1$ for $\alpha^* = \beta^* = 1$, while $a'_{\alpha^* \beta^*} = 0$ for $(\alpha^*, \beta^*) \neq (1, 1)$. From (3) we have

$$(4) \quad A = B_{\alpha T}^{-1} | \alpha | A' | \beta | C_{\sigma\beta}^{-1}.$$

If we let $B_{\alpha T}^{-1} = (b^{\alpha T})$, $C_{\sigma\beta}^{-1} = (c^{\sigma\beta})$, then by (4) $A = (b^{1T} c^{\sigma 1})$, whence $A = (b^{1T}) \times (c^{\sigma 1})$. This proves the theorem.

If the matrix A of a multilinear form G satisfies the relation

$$A = B \times C,$$

it also satisfies the relation

$$A = kB \times (1/k) C,$$

where k is a scalar. Hence the form G factors uniquely only if a further restriction is placed upon G . We therefore have

THEOREM 11. *If the leading coefficient of a multilinear form G is unity, the multilinear form is factorable into 2 unique factors with leading coefficients unity if and only if the T -rank of G^T is unity, where T is the total partition of the indices of one of the factors of G .*

¹⁶ F. L. Hitchcock, Ibid.

¹⁷ By ranks, partitions, ... of G we mean the ranks, partitions, ... of the matrix A associated with G .

The factorization property of Theorem 10 gives

THEOREM 12. *Let the partitions $\rho_1, \dots, \rho_n$ of a matrix $A = (a_{T\sigma})$ of composition on T satisfy the relations $\rho_i = T_i \sigma_i$ for $i = 1, 2, \dots, n$, where $T = T_1 T_2 \dots T_n$, $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$. If the $\rho_1, \rho_2, \dots, \rho_n$ -ranks of A are unity, the composition with A on T is the same as a set of compositions $(b_{T_1 \sigma_1}), (c_{T_1 \sigma_2}), \dots$, on the partitions $T_1, T_2, \dots$, respectively.*

By Theorems 1 and 10 and Cor. 2 of Theorem 7 we have

THEOREM 13. *If the T -rank of a matrix $A = (a_{T\sigma})$ is unity, the α and β -ranks of A are equal, where α and β simply cover T .*

By Theorem 3 we have

THEOREM 14. *If the $T_1, \dots, T_s$ -ranks of a matrix $A = (a_{T_1 \dots T_s \sigma})$ are all unity or zero, the T -rank of A is unity or zero, where $T_1, \dots, T_s$ simply cover T .*

10. Properties of identity matrices. Identity matrices satisfy certain factorization properties. We have

THEOREM 15. *If $\delta = (\delta_{T_1 T_2})$ is an identity matrix on T_1, T_2 , where δ is of class $2r$, $T_1 = i_1 \dots i_r$, $T_2 = j_1 \dots j_r$, and if $\delta_1 = (\delta_{i_1 j_1}), \dots, \delta_r = (\delta_{i_r j_r})$ are identity matrices on $i_1, j_1, \dots$, then δ is the zero-composite of r scalar matrices $s_1 \times \delta_1, \dots, s_r \times \delta_r$, where $s_1 \dots s_r = 1$.*

Evidently $\delta = \delta_1 \times \delta_2 \times \dots \times \delta_r$. If we let $(a_{i_1 j_1}), (a_{i_2 j_2}), \dots, (a_{i_r j_r})$ be the general factors of δ , we have

$$(5) \quad a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_r j_r} = \delta_{i_1 \dots i_r j_1 \dots j_r}.$$

Solution of (5) gives the scalar matrices of the theorem.

If from a partition T we delete all partitions which simply cover T , we shall say that we obtain a vacuous partition of order 1. We are now able to prove

THEOREM 16. *The T -rank of δ , where δ is an identity matrix on $i_1 \dots i_n, j_1 \dots j_n$, is equal to the order of the subset of T obtained by deleting all partitions of the type $i_t j_t$ (t not summed) from T .*

Consider the T -rank of $\delta = (\delta_{i_1 i_2 \dots i_n j_1 j_2 \dots j_n})$. Let the indices in T obtained after deleting all partitions of the type $i_t j_t$ be given by $i_u, i_w, \dots, j_\alpha, j_\beta, \dots$. Let the partition T' be obtained from T by replacing $j_\alpha, j_\beta, \dots$ by $i_\alpha, i_\beta, \dots$. By the symmetry of δ the T and T' -ranks of δ are equal. By Theorems 8 and 15 the T' -rank of δ is equal to the T_1^1 -rank of δ' , where δ' is an identity matrix on T_1^1, T_1^2 ; $T_1^1 = i_u i_w \dots i_\alpha i_\beta \dots$, $T_1^2 = j_u j_w \dots j_\alpha j_\beta \dots$. Let Π_1' be the order of T_1^1 . The T_1^1 -rank of δ' is Π_1' , which proves the theorem.

11. Ranks of inverse matrices. From the results of §10 we shall prove

THEOREM 17. *If $A = (a_{T_1 T_2})$ has an inverse on T_1, T_2 , then A is non-singular on any partition T contained in T_1 or T_2 .*

Let $\delta_1 = (\delta_{T_1 T_1'})$ be an identity matrix on T_1, T_1' . By definition

$$A \mid T_2 \mid A_{T_1 T_2}^{-1} = \delta_1.$$

By Theorem 5 the T -rank of δ_1 is equal to the T -rank of A , where T is contained in T_1 . By Theorem 16 δ_1 is non-singular on T .

12. Canonical multilinear forms. By Theorem 6 and well-known theory on canonical bilinear forms we have

THEOREM 18. *Let a multilinear form F be given by*

$$a_{i_1 \dots i_n} x_{i_1}^1 \dots x_{i_n}^n.$$

Let the partitions $T_{\alpha\beta} = i_\alpha i_\beta$, $T_{\gamma\delta} = i_\gamma i_\delta$, $\dots$, $T_{\xi\chi} = i_\xi i_\chi$ be mutually exclusive partitions of the indices of F . Let the i_ν -rank of F be denoted by r_ν for $\nu = 1, 2, \dots, n$. Let $i_\rho, \dots, i_\mu$ be such that $T_{\alpha\beta}, T_{\gamma\delta}, \dots, T_{\xi\chi}, i_\rho, \dots, i_\mu$ simply cover the total partition of the indices of F . If the $T_{\alpha\beta}, T_{\gamma\delta}, \dots, T_{\xi\chi}, i_\rho, \dots, i_\mu$ -ranks of F are unity, F is equivalent under non-singular linear transformations in the field of F to the form

$$(6) \quad K = b_{T_{\alpha\beta} T_{\gamma\delta} \dots T_{\xi\chi} i_\rho \dots i_\mu} y_{i_\rho}^{\rho} \dots y_{i_\mu}^{\mu} w_{i_\alpha} z_{i_\beta} u_{i_\gamma} v_{i_\delta} \dots x_{i_\xi} t_{i_\chi},$$

where $b_{T_{\alpha\beta} \dots T_{\xi\chi} i_\rho \dots i_\mu}^* = 1$ for $i_\rho = \dots = i_\mu = 1$; $i_\alpha = i_\beta = 1, 2, \dots, r_\alpha$; $i_\gamma = i_\delta = 1, 2, \dots, r_\gamma$; $\dots$; $i_\xi = i_\chi = 1, 2, \dots, r_\xi$; $b_{T_{\alpha\beta} \dots T_{\xi\chi} i_\rho \dots i_\mu}^* = 0$ for all other values of the subscripts.

If we take the matrix L of K to have the same range for each index as the corresponding index of F , then L is an identity matrix or an identity matrix bordered by zeros. Since K is determined by the $i_1, \dots, i_n, T_{\alpha\beta}, T_{\gamma\delta}, \dots, T_{\xi\chi}$ -ranks of F , the remaining ranks of F are determined by these. Let σ be the partition obtained from the partition T of F by deleting all subsets of the type $T_{\alpha\beta}, \dots, T_{\xi\chi}, i_\rho, \dots, i_\mu$. It is readily shown that the T -rank of F is equal to the σ -rank of F . By Theorem 18 the σ -rank of F is equal to the product of the $i_j, i_f, \dots, i_g$ -ranks of F , where $i_j, i_f, \dots, i_g$ simply cover σ .

When the matrix A of F is non-singular on all of its indices and $T_{\alpha\beta}, T_{\gamma\delta}, \dots, T_{\xi\chi}$ simply cover the partition of all of the indices of F , then L is an identity matrix, and the matrices of transformation to identity form a matrix $A_{T_1 T_2}^{-1}$ such that

$$(7) \quad A \mid T_2 \mid A_{T_1 T_2}^{-1} = \delta; A \mid T_1 \mid A_{T_1 T_2}^{-1} = \delta,$$

where $L = \delta = (\delta_{T_1 T_2})$ is an identity matrix on T_1, T_2 .

The canonical forms (6) are the only forms which are completely characterized by 2-way ranks as invariants for non-singular linear transformations.

13. Note on the lemma. By the Lemma and the equivalence of a set of linear transformations to a single composition we are able to extend the entire theory of the invariants of 2-way matrices with slight modifications to n -way matrices where $n \geq 2$. For example, from the theory of pairs of bilinear forms¹⁸ we have at once.

¹⁸ L. E. Dickson, *Modern Algebraic Theories*, p. 113 ff.

THEOREM 19. *Let 2 pairs of multilinear forms P_1, P_2 have the matrices $[A = (a_{T_1T_2}), B = (b_{T_1T_2})]$ and $[C = (c_{T_1T_2}), D = (d_{T_1T_2})]$ respectively. Let two 2-way matrices E, F be given by*

$$E = (e_{T_1T_2}) = (\rho a_{T_1T_2} + \sigma b_{T_1T_2})$$

$$F = (f_{T_1T_2}) = (\rho c_{T_1T_2} + \sigma d_{T_1T_2}).$$

If the determinants of E and F are not identically zero, and if the pairs of multilinear forms P_1, P_2 are equivalent under non-singular linear transformations, then the matrices E and F possess the same invariant factors.

Since n -way matrices, $n > 2$, may be displayed in 2-way form in several ways there may exist several sets of invariant factors for a pair of forms.

14. Space determinants. Hitchcock¹⁹ has shown that certain space ranks of a polyadic P are invariant under non-singular linear transformations by exhibiting multiple concomitants which have as coefficients the minors of a space determinant associated with P . Using theorems on the space determinants of $A \mid T \mid B$, we shall prove relations between the space ranks of the matrices $A \mid T \mid B, A$, and B which include the ranks and the invariant properties established by Hitchcock.

We shall need a few notations. Let $(\alpha \cdots \gamma)$ be a substitution on the set of numbers $(1 \cdots n)$, where $\alpha, \cdots, \gamma = 1, \cdots, n$ in some order. Let $[\alpha \cdots \gamma]$ be $+$ or $-$ according as $(\alpha \cdots \gamma)$ is an even or odd substitution. Let $(\epsilon \cdots \mu)$, S be further substitutions on $(1 \cdots n)$. Let the product substitutions $S(\alpha \cdots \gamma)$, $S(\epsilon \cdots \mu)$ be represented by $(\alpha' \cdots \gamma'), (\epsilon' \cdots \mu')$ respectively. Evidently

$$(8) \quad [\alpha \cdots \gamma] [\epsilon \cdots \mu] = [\alpha' \cdots \gamma'] [\epsilon' \cdots \mu'].$$

If a p -way determinant D of order n is the expansion of $|a_{i_1 i_2 \dots i_p}|$ with the indices $i_1, \dots, i_r$ signant,²⁰ by the notation of (8)

$$(9) \quad D = \sum_{\substack{(\dot{i}_{11} \dots \dot{i}_{1n}) \\ \vdots \\ (\dot{i}_{p-1,1} \dots \dot{i}_{p-1,n})}} [\dot{i}_{11} \dots \dot{i}_{1n}] [\dot{i}_{21} \dots \dot{i}_{2n}] \cdots [\dot{i}_{r1} \dots \dot{i}_{rn}] \\ a_{i_{11} \dots i_{p1}} \cdots a_{i_{1n} \dots i_{pn}},$$

where the summation is over all substitutions on $(\dot{i}_{11} \dots \dot{i}_{1n}), \dots, (\dot{i}_{p-1,1} \dots \dot{i}_{p-1,n})$; $(\dot{i}_{11} \dots \dot{i}_{1n}), \dots, (\dot{i}_{p1} \dots \dot{i}_{pn})$ are substitutions on $(1 \cdots n)$. By (8) D is unique if r is even.²¹ Hence we always take r even. The following theorem on extended Cayley products is proved by Rice.²²

¹⁹ F. L. Hitchcock, Multiple invariants and generalized rank of a p -way matrix of tensor, *Journal of Mathematics and Physics*, vol. 7 (1927), pp. 55-63.

²⁰ Defined by L. H. Rice, Introduction to Higher Determinants, *Journal of Mathematics and Physics*, vol. 9 (1930), p. 48.

²¹ Also mentioned by Rice, *Ibid*.

²² L. H. Rice, *Ibid*.

THEOREM 20. Let an n th-order determinant D be given by

$$\left| \sum_{i_p=1}^m a_{i_1 \dots i_p} b_{i_p j_2 \dots j_q} \right|, m \geq n,$$

where an odd number of the i 's and an odd number of the j 's are signant. Let Γ denote an n th-order subset of $1, \dots, m$. Let A_Γ, B_Γ denote n th-order minors of $(a_{i_1 \dots i_p}), (b_{i_p j_2 \dots j_q})$ obtained by letting i_p range over the subset Γ . Let $|A_\Gamma|, |B_\Gamma|$ denote the determinants of A_Γ, B_Γ expanded with i_p signant, and $i_1, \dots, i_{p-1}, j_2, \dots, j_q$ carrying the same signancy as in D . Then

$$D = \sum_{\Gamma} |A_\Gamma| |B_\Gamma|,$$

where Γ varies over the class of all n th order subsets of $1, 2, \dots, m$.

If $n > m$, D in Theorem 20 is zero. We shall also need

THEOREM 21. Let D be an n th-order determinant given by

$$\left| \sum_{i_p=1}^m a_{i_1 \dots i_p} b_{i_p j_2 \dots j_q} \right|,$$

where an even number of the i 's and an even number of the j 's are signant. Let A_Γ denote an n th-order matrix $(a_{i_1 \dots i_{p-1} \rho})$ determined by letting ρ vary over the range Γ , where Γ is an n th-order subset of $1, \dots, m$, repetitions of values allowed. Let $|A_\Gamma|$ be the determinant of A_Γ expanded with ρ non-signant. Similarly, let $|B_\Gamma|$ be the determinant of an n th-order matrix $(b_{\rho j_2 \dots j_q})$ expanded with ρ non-signant. The remaining indices of $|A_\Gamma|$ and $|B_\Gamma|$ carry the same signancy as in D . Then

$$D = \sum_{\Gamma} 1/k_\Gamma |A_\Gamma| |B_\Gamma|,$$

where Γ varies over all n th-order subsets of $1, 2, \dots, m$, repetitions of values allowed, and the numbers of k_Γ are integers.

It is no restriction to assume that $i_1, i_2, \dots, i_s, j_2, j_3, \dots, j_t$ are signant in D while the remaining indices are non-signant. By (9) D can be written as

$$(10) \quad \sum_{i_{p1} \dots i_{pn}} \left(\sum_{\substack{(i_{11} \dots i_{1n}) \\ \vdots \\ (i_{p-1,1} \dots i_{p-1,n})}} [i_{11} \dots i_{1n}] \dots [i_{s1} \dots i_{sn}] a_{i_{11} \dots i_{p1}} \dots a_{i_{1n} \dots i_{pn}} \right) \\ \left(\sum_{\substack{(j_{21} \dots j_{2n}) \\ \vdots \\ (j_{q-1,1} \dots j_{q-1,n})}} [j_{21} \dots j_{2n}] \dots [j_{t1} \dots j_{tn}] b_{i_{p1} j_{21} \dots j_{q1}} \dots b_{i_{pn} j_{2n} \dots j_{qn}} \right).$$

Of the factors exhibited in (10), those obtained by letting $i_{p1}, i_{p2}, \dots, i_{pn}$ have distinct values are determinants $|A_\Gamma|, |B_\Gamma|$. The terms in (10) which are

such that $i_{p1} = i_{p2} = \dots = i_{pr}$, $i_{p,r+1} = i_{p,r+2} = \dots = i_{p,r+u}$, $\dots$ can be written $1/k_\Gamma \mid A_\Gamma \mid \mid B_\Gamma \mid$, where $k_\Gamma = r! u! \dots w!$. This proves the theorem.

15. Properties of space ranks under general compositions. Let a matrix A be given by $(a_{T_1 \dots T_\rho T_{\rho+1} \dots T_p})$. Let the partition $T_{\rho+t}$ of A be given by the set of indices $i_{t1} \dots i_{ts_t}$ for $t = 1, \dots, p - \rho$. Let Π be the smallest of the orders of $T_1, \dots, T_\rho$, and let n_{tq} be the order of i_{tq} in $T_{\rho+t}$ for $q = 1, \dots, s_t$, $t = 1, \dots, p - \rho$. Let partitions $\alpha_{\rho+t}$ be given by $\alpha_{\rho+t} = j_{t1} \dots j_{ts_t}$, $t = 1, \dots, p - \rho$, where the index j_{tq} ranges over $1, \dots, n_{tq}$, $n_{tq} + 1, \dots, 2n_{tq}, \dots, \Pi n_{tq}$. Let further a matrix A' be given by $(a'_{T_1 \dots T_\rho \alpha_{\rho+1} \dots \alpha_p})$ where $a'_{T_1 \dots T_\rho \alpha_{\rho+1} \dots \alpha_p} = a_{T_1 \dots T_\rho T_{\rho+1} \dots T_p}$ when every index $j_{tq} \equiv i_{tq} \pmod{n_{tq}}$ for $q = 1, \dots, s_t$, $t = 1, \dots, p - \rho$. The matrix A' will be called the $(T_1 \dots T_\rho, T_{\rho+1} \dots T_p)$ -derivate of A .

Examples. In the case of a rectangular two-way matrix $B = (b_{ij})$ of order n by m the (ij) -derivate of B is B itself; the (i, j) -derivate of B (which cannot be used in the treatment of ranks below) is given by

$$(B_1 B_2 \dots B_n) = \begin{bmatrix} b_{11} \dots b_{1m} & b_{11} \dots b_{1m} \dots b_{1m} \\ \vdots & \\ b_{n1} \dots b_{nm} & b_{11} \dots b_{nm} \dots b_{nm} \end{bmatrix},$$

where $B_1 = B_2 = \dots = B_n = B$.

We shall need the

DEFINITION. The p -way rank²³ $r(T_1 \dots T_\rho, T_{\rho+1} \dots T_p)$, $\rho \geq 2$, of $A = (a_{T_1 \dots T_p})$ is the order of the largest non-vanishing minor of the $(T_1 T_2 \dots T_\rho, T_{\rho+1} \dots T_p)$ -derivate of A , where $T_1, T_2, \dots, T_\rho$ are signant and $\alpha_{\rho+1}, \alpha_{\rho+2}, \dots, \alpha_p$ are non-signant.

If all partitions are signant we have $r(T_1 \dots T_p)$.

Let $T_1^{s_1}, \dots, T_p^{s_p}$ be partitions obtained from $T_1, \dots, T_p$ by making the substitutions $S_1, \dots, S_p$ on the indices of $T_1, \dots, T_p$ respectively. Evidently the rank $r(T_1^{s_1} \dots T_\rho^{s_\rho}, T_{\rho+1}^{s_{\rho+1}} \dots T_p^{s_p})$ is equal to the rank $r(T_1 \dots T_\rho, T_{\rho+1} \dots T_p)$ of A .

We shall now prove

THEOREM 22. Let three matrices A, B, C be given by $(a_{T_1 \dots T_\sigma})$, $(b_{T_\sigma \gamma_1 \dots \gamma_\omega})$, $A \mid T_\sigma \mid B$, respectively. Let $\delta_1 = T_{\epsilon+1} \gamma_{\lambda+1}, \dots, \delta_{\nu+\chi} = T_{\epsilon+\nu+\chi} \gamma_{\lambda+\nu+\chi}$. Let the matrix $\delta = (\delta_{T_{\epsilon+s} T'_{\epsilon+s}}) \times \dots \times (\delta_{T_{\epsilon+\nu+\chi} T'_{\epsilon+\nu+\chi}})$, where $(\delta_{T_{\epsilon+s} T'_{\epsilon+s}})$ is an identity matrix on $T_{\epsilon+s}, T'_{\epsilon+s}$ for $s = 1, \dots, \nu + \chi$. Let partitions $\delta'_1 = T'_{\epsilon+1} \gamma_{\lambda+1}, \dots, \delta'_{\nu+\chi} = T'_{\epsilon+\nu+\chi} \gamma_{\lambda+\nu+\chi}$. Let further $\rho = T_{\epsilon+1} \dots T_{\epsilon+\nu+\chi} T_\sigma$. Then

1. if $\epsilon, \nu + \lambda$ are both odd the rank $r(\delta_1 \dots \delta_\nu, T_1 \dots T_\epsilon \gamma_1 \dots \gamma_\lambda, T_{\epsilon+\nu+\chi+1} \dots T_{\sigma-1} \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta_{\nu+1} \dots \delta_{\nu+\chi})$ of C is less than or equal to the rank $r(T_1 \dots$

²³ The ranks defined by Hitchcock, Op. Cit., are the special cases of $r(T_1 \dots T_\rho, T_{\rho+1} \dots T_p)$ where $\rho = p$ or $p - 1$.

$T_\epsilon \rho, T_{\epsilon+\nu+\chi+1} \cdots T_{\sigma-1}$ of A and the rank $r(\delta'_1 \cdots \delta'_\nu \gamma_1 \cdots \gamma_\lambda \rho, \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta'_{\nu+1} \cdots \delta'_{\nu+\chi})$ of the matrix $B \times \delta$;

2. if $\epsilon, \nu + \lambda$ are both even, and $\epsilon \geq 2, \nu + \lambda \geq 2$, the rank $r(\delta_1 \cdots \delta_\nu T_1 \cdots T_\epsilon \gamma_1 \cdots \gamma_\lambda, T_{\epsilon+\nu+\chi+1} \cdots T_{\sigma-1} \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta_{\nu+1} \cdots \delta_{\nu+\chi})$ of C is less than or equal to the rank $r(T_1 \cdots T_\epsilon, \rho T_{\epsilon+\nu+\chi+1} \cdots T_{\sigma-1})$ of A and the rank $r(\delta'_1 \cdots \delta'_\nu \gamma_1 \cdots \gamma_\lambda, \rho \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta'_{\nu+1} \cdots \delta'_{\nu+\chi})$ of $B \times \delta$.

The rank $r(\delta_1 \cdots \delta_\nu T_1 \cdots T_\epsilon \gamma_1 \cdots \gamma_\lambda, T_{\epsilon+\nu+\chi+1} \cdots T_{\sigma-1} \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta_{\nu+1} \cdots \delta_{\nu+\chi})$ of C is evidently the same as the rank $r(\delta'_1 \cdots \delta'_\nu T_1 \cdots T_\epsilon \gamma_1 \cdots \gamma_\lambda, T_{\epsilon+\nu+\chi+1} \cdots T_{\sigma-1} \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta'_{\nu+1} \cdots \delta'_{\nu+\chi})$ of C_1 , where

$$C_1 = A \mid \rho \mid B \times \delta.$$

Let $|M|_q$ be a q th-order minor of the $(\delta'_1 \cdots \delta'_\nu T_1 \cdots T_\epsilon \gamma_1 \cdots \gamma_\lambda, T_{\epsilon+\nu+\chi+1} \cdots T_{\sigma-1} \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta'_{\nu+1} \cdots \delta'_{\nu+\chi})$ -derivate of C_1 expanded with $\delta'_1, \cdots, \delta'_\nu, T_1, \cdots, T_\epsilon, \gamma_1, \cdots, \gamma_\lambda$ signant. If $\epsilon, \nu + \lambda$ are odd, we apply Theorem 20 to $|M|_q$; if $\epsilon, \nu + \lambda$ are even, we apply Theorem 21, whence Theorem 22 is proved.

The ordering of the signant and non-signant partitions used in Theorem 22 is no restriction of the problem since the partitions can be chosen at random from the indices of A . The ranks of C covered by this theorem compose the class of all possible ranks of C for which $\epsilon, \nu + \lambda \geq 1$.

In our discussion above we defined the p -way rank of a matrix A in terms of the minors of a derivate of A where these minors possessed the property that at least two partitions were signant. By constructing new derivates of A we shall now define p -way ranks of A in terms of minors which have no partitions signant. These ranks will be called *non-signant ranks*. By means of these ranks we shall be able to treat the ranks of C occurring in Theorem 22 for which $\epsilon = 0$ or $\nu + \lambda = 0$.

Let a matrix A be given by $(a_{T_1 \cdots T_p})$ as before. Let $\mu_1, \cdots, \mu_p$ be arbitrary positive integers. Let the partition T_t of A be given by the set of indices $i_{t1} \cdots i_{ts_t}$ for $t = 1, \cdots, p$, and let n_{tq} be the order of i_{tq} for $q = 1, \cdots, s_t, t = 1, \cdots, p$. Let partitions α_t be given by $\alpha_t = j_{t1} \cdots j_{ts_t}$ for $t = 1, \cdots, p$. Let a matrix $A' = (a'_{\alpha_1 \cdots \alpha_p})$ satisfy the following properties;

1. The index j_{tq} ranges over $1, \cdots, n_{tq}, n_{tq} + 1, \cdots, 2n_{tq}, \cdots, \mu_t n_{tq}$ for $q = 1, \cdots, s_t, t = 1, \cdots, p$;

2. $a'_{\alpha'_1 \cdots \alpha'_p} = a_{T_1 \cdots T_p}$ when every index $j_{tq} \equiv i_{tq} \pmod{n_{tq}}$ for $q = 1, \cdots, s_t, t = 1, \cdots, p$.

The matrix A' so defined is called the $(T_1 \cdots T_p; \mu_1 \cdots \mu_p)$ -derivate of A .

Example. The $(ij; nm)$ -derivate of $B = (b_{ij})$ is given by

$$\begin{bmatrix} B_{11} & \cdots & B_{1m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ B_{n1} & \cdots & B_{nm} \end{bmatrix},$$

where $B_{\rho\sigma} = B$ for $\rho = 1, \cdots, n; \sigma = 1, \cdots, m$.

Let a minor $|M|$ of the $(T_1 \cdots T_p; \mu_1 \cdots \mu_p)$ -derivate of $A = (a_{T_1 \dots T_p})$ satisfy the properties:

1. It is of order less than or equal to α .
2. It has no partitions signant.

DEFINITION. The p -way rank $r(T_1 \cdots T_p; \mu_1 \cdots \mu_p; \alpha)$ of $A = (a_{T_1 \dots T_p})$ is the order of the largest non-vanishing minor $|M|$ satisfying the properties 1 and 2.

Let $T_1^{s_1}, \dots, T_p^{s_p}$ be defined as above. In the case of non-signant ranks it is again evident the rank $r(T_1^{s_1} \cdots T_p^{s_p}; \mu_1 \cdots \mu_p; \alpha)$ of A is equal to the rank $(T_1 \cdots T_p; \mu_1 \cdots \mu_p; \alpha)$ of A .

In the remainder of the paper a minor $|M|_q$ of order q of the $(T_1 \cdots T_p, T_{p+1} \cdots T_p)$ -derivate of a matrix $A = (a_{T_1 \dots T_p})$ will denote a minor of this derivate obtained by letting $T_1, \dots, T_p$ be signant partitions, and $T_{p+1}, \dots, T_p$ non-signant partitions in the expansion of this minor. It will also be understood that a minor $|M|_q$ of order q of the $(T_1 \cdots T_p; \mu_1 \cdots \mu_p)$ -derivate of A will denote a minor of this derivate obtained by letting all partitions be non-signant.

In the case where $p = 1$, so that $r(T_1 \cdots T_p; \mu_1 \cdots \mu_p; \alpha)$ of A contains only one partition we have the evident

THEOREM 23. If at least one element of $A = (a_T)$ is not zero, and if $\alpha \geq \mu$, the rank $r(T; \mu; \alpha)$ of A is greater than or equal to μ . If only one element is not zero and $\alpha \geq \mu$, $r(T; \mu; \alpha)$ of A is μ . If $\mu \geq \alpha$, in either case $r(T; \mu; \alpha)$ of A is α .

We shall now prove

THEOREM 24. Let three matrices A, B, C be given by $(a_{T_1 \dots T_\sigma}), (b_{T_\sigma \gamma_1 \dots \gamma_\omega}), A | T_\sigma | B$, respectively. Let $\delta_1 = T_{\epsilon+1} \gamma_1, \dots, \delta_\chi = T_{\epsilon+\chi} \gamma_\chi$. Let the matrix $\delta = (\delta_{T_{\epsilon+1} T'_{\epsilon+1}}, \dots, \delta_{T_{\epsilon+\chi} T'_{\epsilon+\chi}})$, where $(\delta_{T_{\epsilon+s} T'_{\epsilon+s}})$ is an identity matrix on $T_{\epsilon+s}, T'_{\epsilon+s}$, for $s = 1, \dots, \chi$. Let $\delta'_1 = T'_{\epsilon+1} \gamma_1, \dots, \delta'_\chi = T'_{\epsilon+\chi} \gamma_\chi$. Let further $\rho = T_{\epsilon+1} \cdots T_{\epsilon+\chi} T_\sigma$. Then the rank $r(T_1 \cdots T_\epsilon, T_{\epsilon+\chi+1} \cdots T_{\sigma-1} \gamma_{\chi+1} \cdots \gamma_\omega \delta_1 \cdots \delta_\chi, \epsilon \geq 2, \text{ of } C \text{ is less than or equal to the rank } r(T_1 \cdots T_\epsilon, T_{\epsilon+\chi+1} \cdots T_{\sigma-1} \rho) \text{ of } A \text{ and } r(\rho \gamma_{\chi+1} \cdots \gamma_\omega \delta'_1 \cdots \delta'_\chi; \mu \cdots \mu; \mu) \text{ of } B \times \delta, \text{ where } \mu \text{ is the smallest of the orders of } T_1, \dots, T_\epsilon$.

The rank $r(T_1 \cdots T_\epsilon, T_{\epsilon+\chi+1} \cdots T_{\sigma-1} \gamma_{\chi+1} \cdots \gamma_\omega \delta_1 \cdots \delta_\chi)$ of C is equal to the rank $r(T_1 \cdots T_\epsilon, T_{\epsilon+\chi+1} \cdots T_{\sigma-1} \gamma_{\chi+1} \cdots \gamma_\omega \delta'_1 \cdots \delta'_\chi)$ of C_1 where

$$C_1 = A | \rho | B \times \delta.$$

We now apply Theorem 21 to the minors of the $(T_1 \cdots T_\epsilon, T_{\epsilon+\chi+1} \cdots T_{\sigma-1} \gamma_{\chi+1} \cdots \gamma_\omega \delta'_1 \cdots \delta'_\chi)$ -derivate of C_1 .

We have at once the

COROLLARY. Let matrices A, B, C be given as in Theorem 24. Let $\delta_1, \dots, \delta_{\nu+\chi}, \delta'_1, \dots, \delta'_{\nu+\chi}, \delta, \rho$ be given as in Theorem 22 except that $\epsilon = 0$. Then the rank $r(\delta_1 \cdots \delta_\nu \gamma_1 \cdots \gamma_\lambda, T_{\nu+\chi+1} \cdots T_{\sigma-1} \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta_{\nu+1} \cdots \delta_{\nu+\chi}, \nu + \lambda > 2, \text{ of } C \text{ is less than or equal to the rank } r(\delta'_1 \cdots \delta'_\nu \gamma_1 \cdots \gamma_\lambda, \rho \gamma_{\lambda+\nu+\chi+1} \cdots \gamma_\omega \delta'_{\nu+1} \cdots \delta'_{\nu+\chi}) \text{ of } B \times \delta \text{ and the rank } r(\rho T_{\nu+\chi+1} \cdots T_{\sigma-1}; \mu \mu \cdots \mu; \mu) \text{ of } A \text{ where } \mu \text{ is the smallest of the orders of } \delta_1, \dots, \delta_\nu, \gamma_1, \dots, \gamma_\lambda$.

For non-signant ranks we have

THEOREM 25. *Let three matrices A, B, C be given by $(a_{\tau_1} \dots \tau_\epsilon)$, $(b_{\tau_\sigma} \gamma_1 \dots \gamma_\omega)$, $A \mid T_\sigma \mid B$, respectively. Let $\delta_1 = T_{\epsilon+1} \gamma_{\lambda+1}, \dots, \delta_\nu = T_{\epsilon+\nu} \gamma_{\lambda+\nu}$. Let the matrix $\delta = (\delta_{T_{\epsilon+1} T'_{\epsilon+1}}) \times \dots \times (\delta_{T_{\epsilon+\nu} T'_{\epsilon+\nu}})$ where $(\delta_{T_{\epsilon+s} T'_{\epsilon+s}})$ is an identity matrix on $T_{\epsilon+s}, T'_{\epsilon+s}$ for $s = 1, \dots, \nu$. Further let $\delta'_1 = T'_{\epsilon+1} \gamma_{\lambda+1}, \dots, \delta'_\nu = T'_{\epsilon+\nu} \gamma_{\lambda+\nu}$, and let $\rho = T_{\epsilon+1} \dots T_{\epsilon+\nu} T_\sigma$. Then the rank $r(\delta_1 \dots \delta_\nu T_1 \dots T_\epsilon \gamma_1 \dots \gamma_\lambda; \mu_1 \dots \mu_\nu \mu_{\nu+1} \dots \mu_{\nu+\epsilon} \mu_{\nu+\epsilon+1} \dots \mu_{\nu+\epsilon+\lambda}; \alpha)$ of C is less than or equal to the rank $r(\rho \delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_\lambda; \alpha \mu_1 \dots \mu_\nu \mu_{\nu+\epsilon+1} \dots \mu_{\nu+\epsilon+\lambda}; \alpha)$ of $B \times \delta$ and the rank $r(\rho T_1 \dots T_\epsilon; \alpha \mu_{\nu+1} \dots \mu_{\nu+\epsilon}; \alpha)$ of A .*

To prove Theorem 25 we construct a matrix $C_1 = A \mid \rho \mid B \times \delta$, and apply Theorem 21.

16. The ranks of $B \times \delta$. The ranks of the matrix $B \times \delta = (b_{\tau_\sigma \gamma_1} \dots \gamma_\omega \delta_{T_{\epsilon+1} \dots T'_{\epsilon+1}} \dots \delta_{T_{\epsilon+\nu+\chi} T'_{\epsilon+\nu+\chi}})$, $\nu + \chi > 0$, mentioned in Theorems 22 to 25 are not directly expressible for a general B in terms of the ranks of B . We shall, however, establish a bound property between the ranks of these matrices. We consider several cases.

1. ρ signant. Assume that a q th-order minor $|B|_q$ of the $(T_\sigma \gamma_1 \dots \gamma_{\lambda+\nu}, \gamma_{\lambda+\nu+1} \dots \gamma_\omega)$ -derivate of B is not zero. Let a minor $|B \times \delta|_q$ of order q be obtained from the $(\delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_{\lambda+\rho}, \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ -derivate of $B \times \delta$ by letting $T_\sigma, \gamma_1, \dots, \gamma_\omega$ have the same ranges as in $|B|_q$, and choose the ranges of $\delta'_1, \dots, \delta'_{\nu+\chi}$, such that $T_{\epsilon+i} = T'_{\epsilon+i} = 1$ for $i = 1, \dots, \nu + \chi$. Evidently

$$|B \times \delta|_q = |B|_q.$$

Hence the rank $r(\delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_{\lambda+\rho}, \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of $B \times \delta$ is greater than or equal to the rank $r(T_\sigma \gamma_1 \dots \gamma_{\lambda+\nu}, \gamma_{\lambda+\nu+1} \dots \gamma_\omega)$ of B .

1. In special cases a better inequality than that of Case 1 can be obtained as follows. Let μ be the smallest of the orders of $T_{\epsilon+1}, \dots, T_{\epsilon+\nu+\chi}$. Assume that $\lambda = 0$ in Case 1 and $\omega = \nu + \chi$, and further that a q th-order minor $|B|_q$ of the $(T_\sigma \gamma_1 \dots \gamma_\nu, \gamma_{\nu+1} \dots \gamma_{\nu+\chi})$ -derivate of B is not zero. Let a minor $|B \times \delta|_{\mu q}$ of order μq be obtained from the $(\delta'_1 \dots \delta'_\nu \rho, \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ -derivate of B by letting $T'_{\epsilon+i} = 1, 2, \dots, \mu$ for $i = 1, 2, \dots, \nu + \chi$; $(T_{\epsilon+1} \dots T_{\epsilon+\nu+\chi}) = (1 \dots 1), (2 \dots 2), \dots, (\mu \dots \mu)$; and $T_\sigma, \gamma_1, \dots, \gamma_{\nu+\chi}$ have the same ranges as in $|B|_q$. Then

$$|B \times \delta|_{\mu q} = |B|_{\mu q},$$

whence the rank $r(\delta'_1 \dots \delta'_\nu \rho, \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of $B \times \delta$ is greater than or equal μr_1 , where $r_1 = r(T_\sigma \gamma_1 \dots \gamma_\nu, \gamma_{\nu+1} \dots \gamma_{\nu+\chi})$ of B .

2. ρ non-signant; $\nu + \lambda \geq 2$. Applying the same argument as in Case 1 using now the $(\gamma_1 \dots \gamma_{\lambda+\nu}, T_\sigma \gamma_{\lambda+\nu+1} \dots \gamma_\omega)$ -derivate of B , and the $(\delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_\lambda, \rho \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ -derivate of $B \times \delta$, we can show that the rank $r(\delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_\lambda, \rho \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of $B \times \delta$ is greater than or equal to the rank $r(\gamma_1 \dots \gamma_{\lambda+\nu}, T_\sigma \gamma_{\lambda+\nu+1} \dots \gamma_\omega)$ of B .

2₁. By methods similar to Case 1₁ we can show that the rank $r(\delta'_1 \cdots \delta'_\nu, \rho \delta'_{\nu+1} \cdots \delta'_{\nu+\chi})$ of $B \times \delta$ is greater than or equal to μr_1 , where $r_1 = r(\gamma_1 \cdots \gamma_\nu, T_\sigma \gamma_{\nu+1} \cdots \gamma_{\nu+\chi})$ of B , and μ is the smallest of the orders of $T_{\epsilon+1}, \cdots, T_{\epsilon+\nu+\chi}$.

3. No partitions signant. Let $\delta'_1 = T'_1 \gamma_1, \cdots, \delta'_\chi = T'_\chi \gamma_\chi, \rho = T_1 \cdots T_\chi T_\sigma$. Assume that $|B|_q \neq 0$, where $|B|_q$ is a minor of order q of the $(T_\sigma \gamma_{\chi+1} \cdots \gamma_\omega \gamma_1 \cdots \gamma_\chi; \mu \mu_{\chi+1} \cdots \mu_\omega \mu_1 \cdots \mu_\chi)$ -derivate of B . Let a minor $|B \times \delta|_q$ of order q of the $(\rho \gamma_{\chi+1} \cdots \gamma_\omega \delta'_1 \cdots \delta'_\chi; \mu \mu_{\chi+1} \cdots \mu_\omega \mu_1 \cdots \mu_\chi)$ -derivate of $B \times \delta$ be obtained by letting $T_\sigma, \gamma_1, \cdots, \gamma_\omega$ have the same ranges as in $|B|_q$, and $T'_{\epsilon+i} = T_{\epsilon+i} = 1$ for $i = 1, \cdots, \chi$. Then

$$|B \times \delta|_q = |B|_q.$$

Hence the rank $r(\rho \gamma_{\chi+1} \cdots \gamma_\omega \delta'_1 \cdots \delta'_\chi; \mu \mu_{\chi+1} \cdots \mu_\omega \mu_1 \cdots \mu_\chi; \alpha)$ of $B \times \delta$ is greater than or equal to the rank $r(T_\sigma \gamma_{\chi+1} \cdots \gamma_\omega \gamma_1 \cdots \gamma_\chi; \mu \mu_{\chi+1} \cdots \mu_\omega \mu_1 \cdots \mu_\chi; \alpha)$ of B .

3₁. A better inequality for special cases of 3 can be obtained as follows. Assume that $|B|_q \neq 0$, where $|B|_q$ is a minor of order q of the $(T_\sigma \gamma_1 \cdots \gamma_\chi; \mu \mu_1 \cdots \mu_\chi)$ -derivate of B . Let a minor $|B \times \delta|_{\Pi q}$ of order Πq of the $(\rho \delta'_1 \cdots \delta'_\chi; \Pi \mu, \Pi \mu_1, \cdots, \Pi \mu_\chi)$ -derivate of $B \times \delta$ be obtained by letting $T_\sigma, \gamma_1, \cdots, \gamma_\chi$ have the same ranges as in $|B|_q$; also $T'_i = 1, \cdots, \Pi$ for $i = 1, \cdots, \chi$, and $(T_1 \cdots T_\chi) = (1 \cdots 1), \cdots, (\Pi \cdots \Pi)$. Π is taken as an arbitrary integer greater than zero and less than the smallest of the orders of $T_1, \cdots, T_\chi$. Then

$$|B \times \delta|_{\Pi q} = |B|_{\frac{\Pi}{q}}.$$

Therefore the rank $r(\rho \delta'_1 \cdots \delta'_\chi; \Pi \mu, \Pi \mu_1, \cdots, \Pi \mu_\chi; \Pi \alpha)$ of $B \times \delta$ is greater than or equal to Πr_1 , where $r_1 = r(\rho \gamma_1 \cdots \gamma_\chi; \mu \mu_1 \cdots \mu_\chi, \alpha)$ of B .

This completes the treatment of the ranks of $B \times \delta$ mentioned in Theorems 22 to 25.

17. Space ranks of $A \times B$. The following theorem is given by Hitchcock.²⁴

THEOREM 26. Let $A = (a_{i1} \cdots i_p), B = (b_{j1} \cdots j_q)$, and $C = A \times B$, where A and B are cube matrices. If $|C|$ is the determinant of C developed with an even number of i 's and j 's signant, then

$$|C| = n! |A| |B|,$$

where $|A|$ and $|B|$ are determinants of A and B with the same signancy as in C .

Theorem 26 implies

THEOREM 27. Let $A = (a_{T_1} \cdots T_p), B = (b_{\gamma_1} \cdots \gamma_\omega), C = A \times B$. If s, t are both even and greater than or equal to 2, the rank $r(T_1 \cdots T_s \gamma_1 \cdots \gamma_t, T_{s+1} \cdots T_p \gamma_{t+1} \cdots \gamma_\omega)$ of C is equal to the smaller of the ranks $r(T_1 \cdots T_s, T_{s+1} \cdots T_p)$ and $r(\gamma_1 \cdots \gamma_t, \gamma_{t+1} \cdots \gamma_\omega)$ of A and B respectively.

If s and t are both odd the rank $r(T_1 \cdots T_s \gamma_1 \cdots \gamma_t, T_{s+1} \cdots T_p \gamma_{t+1} \cdots \gamma_\omega)$

²⁴ F. L. Hitchcock, Ordered determinants with applications to polyadics, Journal of Mathematics and Physics, vol. 4 (1925), p. 213.

of C is 1 or 0 according as A and B contain non-zero elements or at least one of these matrices is a zero matrix.

THEOREM 28. Let $A = (a_{T_1 \dots T_p})$, $B = (b_{\gamma_1 \dots \gamma_\omega})$, and $C = A \times B$. Let Π be the smallest of the orders of the partitions $T_1, \dots, T_s$. The rank $r(T_1 \dots T_s, T_{s+1} \dots T_p \gamma_1 \dots \gamma_\omega)$ of C is equal to the smaller of the ranks $r(T_1 \dots T_s, T_{s+1} \dots T_p)$ of A and $r(\gamma_1 \dots \gamma_\omega; \Pi \dots \Pi; \Pi)$ of B .

Theorem 28 and the following theorem follow from Theorem 26.

THEOREM 29. Let $A = (a_{T_1 \dots T_p})$, $B = (b_{\gamma_1 \dots \gamma_\omega})$, and $C = A \times B$. The rank $r(T_1 \dots T_p \gamma_1 \dots \gamma_\omega; \mu_1 \dots \mu_p \mu_{p+1} \dots \mu_{p+\omega}; \alpha)$ of C is equal to the smaller of the ranks $r(T_1 \dots T_p; \mu_1 \dots \mu_p; \alpha)$ of A and $r(\gamma_1 \dots \gamma_\omega; \mu_{p+1} \dots \mu_{p+\omega}; \alpha)$ of B .

We shall now prove

THEOREM 30. Let $A = (a_{T_1 \dots T_p})$, $B = (b_{\gamma_1 \dots \gamma_\omega})$, $C = A \times B$. Let $\delta_1 = T_{\epsilon+1} \gamma_{\lambda+1}, \dots, \delta_{\nu+\chi} = T_{\epsilon+\nu+\chi} \gamma_{\lambda+\nu+\chi}$. Further let $\delta = (\delta_{T_{\epsilon+1} T'_{\epsilon+1}}) \times \dots \times (\delta_{T_{\epsilon+\nu+\chi} T'_{\epsilon+\nu+\chi}})$ where $(\delta_{T_{\epsilon+s} T'_{\epsilon+s}})$ is an identity matrix on $T_{\epsilon+s}, T'_{\epsilon+s}$ for $s = 1, \dots, \nu + \chi$, and $\rho = T_{\epsilon+1} \dots T_{\epsilon+\nu+\chi}$, and let $\delta'_1 = T'_{\epsilon+1} \gamma_{\lambda+1}, \dots, \delta'_{\nu+\chi} = T'_{\epsilon+\nu+\chi} \gamma_{\lambda+\nu+\chi}$. Assume that $\epsilon \geq 1, \nu + \lambda \geq 1, \nu + \chi > 0$. Then

1. if $\epsilon, \nu + \lambda$ are both odd, the rank $r(\delta_1 \dots \delta_{\nu} T_1 \dots T_{\epsilon} \gamma_1 \dots \gamma_{\lambda}, T_{\epsilon+\nu+\chi+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_{\omega} \delta_{\nu+1} \dots \delta_{\nu+\chi})$ of C is less than or equal to the ranks $r(T_1 \dots T_{\epsilon} \rho, T_{\epsilon+\nu+\chi+1} \dots T_p)$ of A and $r(\delta'_1 \dots \delta'_{\nu} \gamma_1 \dots \gamma_{\lambda} \rho, \gamma_{\lambda+\nu+\chi+1} \dots \gamma_{\omega} \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of $B \times \delta$.

2. if $\epsilon, \nu + \lambda$ are both even, the rank $r(\delta_1 \dots \delta_{\nu} T_1 \dots T_{\epsilon} \gamma_1 \dots \gamma_{\lambda}, T_{\epsilon+\nu+\chi+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_{\omega} \delta_{\nu+1} \dots \delta_{\nu+\chi})$ of C is less than or equal to the ranks $r(T_1 \dots T_{\epsilon}, \rho T_{\epsilon+\nu+\chi+1} \dots T_p)$ of A and $r(\delta'_1 \dots \delta'_{\nu} \gamma_1 \dots \gamma_{\lambda}, \rho \gamma_{\lambda+\nu+\chi+1} \dots \gamma_{\omega} \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of $B \times \delta$.

The rank $r(\delta_1 \dots \delta_{\nu} T_1 \dots T_{\epsilon} \gamma_1 \dots \gamma_{\lambda}, T_{\epsilon+\nu+\chi+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_{\omega} \delta_{\nu+1} \dots \delta_{\nu+\chi})$ of C is equal to the rank $r(\delta'_1 \dots \delta'_{\nu} T_1 \dots T_{\epsilon} \gamma_1 \dots \gamma_{\lambda}, T_{\epsilon+\nu+\chi+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_{\omega} \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of C_1 , where

$$C_1 = A \mid \rho \mid B \times \delta.$$

According as $\epsilon, \nu + \lambda$ are both odd or both even we apply Theorems 22₁ and 22₂.

For the cases where $\nu + \lambda = 0$ in the rank of C given above in Theorem 30 we prove

THEOREM 31. Let $A = (a_{T_1 \dots T_p})$, $B = (b_{\gamma_1 \dots \gamma_\omega})$, $C = A \times B$. Let $\chi \geq 1$. Let $\delta_1 = T_{\epsilon+1} \gamma_1, \dots, \delta_{\chi} = T_{\epsilon+\chi} \gamma_{\chi}$, and $\delta = (\delta_{T_{\epsilon+1} T'_{\epsilon+1}}) \times \dots \times (\delta_{T_{\epsilon+\chi} T'_{\epsilon+\chi}})$, where $(\delta_{T_{\epsilon+s} T'_{\epsilon+s}})$ is an identity matrix on $T_{\epsilon+s}, T'_{\epsilon+s}$ for $s = 1, \dots, \chi$. Let $\delta'_1 = T'_{\epsilon+1} \gamma_1, \dots, \delta'_{\chi} = T'_{\epsilon+\chi} \gamma_{\chi}$. Further let $\rho = T_{\epsilon+1} \dots T_{\epsilon+\chi}$, and let Π be the smallest of the orders of $T_1, \dots, T_{\epsilon}$. Then the rank $r(T_1 \dots T_{\epsilon}, T_{\epsilon+\chi+1} \dots T_p \gamma_{\chi+1} \dots \gamma_{\omega} \delta_1 \dots \delta_{\chi})$ of C is less than or equal to the ranks $r(T_1 \dots T_{\epsilon}, T_{\epsilon+\chi+1} \dots T_p \rho)$ of A and $r(\rho \gamma_{\chi+1} \dots \gamma_{\omega} \delta'_1 \dots \delta'_{\chi}; \Pi \dots \Pi; \Pi)$ of $B \times \delta$.

The rank $r(T_1 \dots T_{\epsilon}, T_{\epsilon+\chi+1} \dots T_p \gamma_{\chi+1} \dots \gamma_{\omega} \delta_1 \dots \delta_{\chi})$ of C is equal to the rank $r(T_1 \dots T_{\epsilon}, T_{\epsilon+\chi+1} \dots T_p \gamma_{\chi+1} \dots \gamma_{\omega} \delta'_1 \dots \delta'_{\chi})$ of C_1 where

$$C_1 = A \mid \rho \mid B \times \delta.$$

We now apply Theorem 24.

For the cases where $\epsilon = 0$ in the rank of C given in Theorem 30 we prove

THEOREM 32. Let $A = (a_{T_1 \dots T_p})$, $B = (b_{\gamma_1 \dots \gamma_\omega})$, $C = A \times B$. Assume that $\nu + \lambda \geq 2$, $\nu + \chi > 0$. Let $\delta_1 = T_1 \gamma_{\lambda+1}, \dots, \delta_{\nu+\chi} = T_{\nu+\chi} \gamma_{\lambda+\nu+\chi}$, and $\delta = (\delta_{T_1 T'_1}) \times \dots \times (\delta_{T_{\nu+\chi} T'_{\nu+\chi}})$, where $(\delta_{T_s T'_s})$ is an identity matrix on T_s , T'_s for $s = 1, \dots, \nu + \chi$. Further let $\rho = T_1 \dots T_{\nu+\chi}$, and let Π be the smallest of the orders of $\gamma_1, \dots, \gamma_\lambda, \delta_1, \dots, \delta_\nu$. Then the rank $r(\delta_1 \dots \delta_\nu \gamma_1 \dots \gamma_\lambda, T_{\chi+\nu+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta_{\nu+1} \dots \delta_{\nu+\chi})$ of C is less than or equal to the ranks $r(\delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_\lambda \rho, \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of $B \times \delta$ and $r(\rho T_{\chi+\nu+1} \dots T_p; \Pi \dots \Pi; \Pi)$ of A .

The rank $r(\delta_1 \dots \delta_\nu \gamma_1 \dots \gamma_\lambda, T_{\chi+\nu+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta_{\nu+1} \dots \delta_{\nu+\chi})$ of C is equal to the rank $r(\delta'_1 \dots \delta'_\nu \gamma_1 \dots \gamma_\lambda, T_{\chi+\nu+1} \dots T_p \gamma_{\lambda+\nu+\chi+1} \dots \gamma_\omega \delta'_{\nu+1} \dots \delta'_{\nu+\chi})$ of C_1 , where

$$C_1 = A \mid \rho \mid B \times \delta.$$

We apply the Corollary to Theorem 24.

For non-signant ranks we have

THEOREM 33. Let $A = (a_{T_1 \dots T_p})$, $B = (b_{\gamma_1 \dots \gamma_\omega})$, $C = A \times B$. Assume that $\chi \geq 1$. Let $\delta_1 = T_1 \gamma_1, \dots, \delta_\chi = T_\chi \gamma_\chi$, and $\delta = (\delta_{T_1 T'_1}) \times \dots \times (\delta_{T_\chi T'_\chi})$ where $(\delta_{T_s T'_s})$ is an identity matrix on T_s , T'_s for $s = 1, \dots, \chi$. Further let $\rho = T_1 \dots T_\chi$. Then the rank $r(T_{\chi+1} \dots T_p \gamma_{\chi+1} \dots \gamma_\omega \delta_1 \dots \delta_\chi; \mu_1 \dots \mu_{p-\chi} \beta_1 \dots \beta_{\omega-\chi} \xi_1 \dots \xi_\chi; \alpha)$ of C is less than or equal to the ranks $r(\rho T_{\chi+1} \dots T_p; \alpha \mu_1 \dots \mu_{p-\chi}; \alpha)$ of A and $r(\rho \gamma_{\chi+1} \dots \gamma_\omega \delta'_1 \dots \delta'_\chi; \alpha \beta_1 \dots \beta_{\omega-\chi} \xi_1 \dots \xi_\chi; \alpha)$ of $B \times \delta$.

The rank $r(T_{\chi+1} \dots T_p \gamma_{\chi+1} \dots \gamma_\omega \delta_1 \dots \delta_\chi; \mu_1 \dots \mu_{p-\chi} \beta_1 \dots \beta_{\omega-\chi} \xi_1 \dots \xi_\chi; \alpha)$ of C is equal to the rank $r(T_{\chi+1} \dots T_p \gamma_{\chi+1} \dots \gamma_\omega \delta'_1 \dots \delta'_\chi; \mu_1 \dots \mu_{p-\chi} \beta_1 \dots \beta_{\omega-\chi} \xi_1 \dots \xi_\chi; \alpha)$ of C_1 , where

$$C_1 = A \mid \rho \mid B \times \delta.$$

We now apply Theorem 25.

In Theorems 27 to 33 we have established relations between all of the possible space ranks of $A \times B$ and the space ranks of A and $B \times \delta$, where δ is an identity matrix.

18. Space ranks of Scott products. Rice²⁵ has shown that the $p + q + 1$ -way determinant $|C| = |a_{i_1 \dots i_p T} b_{T j_1 \dots j_q}|$, where T is not summed, is equal to the product of the determinants $|A| = |a_{i_1 \dots i_p T}|$, $|B| = |b_{T j_1 \dots j_q}|$, where the i 's and j 's in $|A|$ and $|B|$ have the same signancy as in $|C|$. The T 's of $|A|$ and $|B|$ are both signant or both non-signant if T of $|C|$ is non-signant, while on the other hand one of the T 's of $|A|$ and $|B|$ is non-signant and the other signant if T of $|C|$ is signant. We have at once

THEOREM 34. If $C = (a_{T_1 \dots T_p T} b_{T T_1 \dots T_q})$, where T is not summed, and $A = (a_{T_1 \dots T_p T})$, $B = (b_{T T_1 \dots T_q})$, then

²⁵ L. H. Rice, Op. Cit., p. 65.

1. the rank $r(T_1 \cdots T_t \gamma_1 \cdots \gamma_s T, T_{t+1} \cdots T_p \gamma_{s+1} \cdots \gamma_\omega)$, $t \geq 1$, $s \geq 1$, of C is equal to the smaller of the ranks $r(T_1 \cdots T_t T, T_{t+1} \cdots T_p)$ of A and $r(\gamma_1 \cdots \gamma_s, T \gamma_{s+1} \cdots \gamma_\omega)$ of B if t is odd and s is even; the rank $r(T_1 \cdots T_t \gamma_1 \cdots \gamma_s T, T_{t+1} \cdots T_p \gamma_{s+1} \cdots \gamma_\omega)$, $t \geq 1$, $s \geq 1$, of C is equal to the smaller of the ranks $r(T_1 \cdots T_t, TT_{t+1} \cdots T_p)$ of A and $r(\gamma_1 \cdots \gamma_s T, \gamma_{s+1} \cdots \gamma_\omega)$ of B if s is odd and t is even;

2. the rank $r(T_1 \cdots T_t \gamma_1 \cdots \gamma_s, TT_{t+1} \cdots T_p \gamma_{s+1} \cdots \gamma_\omega)$, $t \geq 1$, $s \geq 1$, of C is equal to the smaller of the ranks $r(T_1 \cdots T_t T, T_{t+1} \cdots T_p)$ of A and $r(\gamma_1 \cdots \gamma_s T, \gamma_{s+1} \cdots \gamma_\omega)$ of B if t, s are odd; the rank $r(T_1 \cdots T_t \gamma_1 \cdots \gamma_s, TT_{t+1} \cdots T_p \gamma_{s+1} \cdots \gamma_\omega)$, $t \geq 1$, $s \geq 1$, of C is equal to the smaller of the ranks $r(T_1 \cdots T_t, TT_{t+1} \cdots T_p)$ of A and $r(\gamma_1 \cdots \gamma_s, T \gamma_{s+1} \cdots \gamma_p)$ of B if t, s are even.

For the cases where $T_1, \dots, T_p$ are all non-signant we have

THEOREM 35. Let $C = (a_{T_1 \dots T_p T} b_{T \gamma_1 \dots \gamma_\omega})$, T not summed, $A = (a_{T_1 \dots T_p T})$, $B = (b_{T \gamma_1 \dots \gamma_\omega})$. Let $s \geq 1$. Let Π be the smallest of the orders of $\gamma_1, \dots, \gamma_s$ and Π_1 the smallest of the orders of $\gamma_1, \dots, \gamma_s, T$. Then

1. the rank $r(\gamma_1 \cdots \gamma_s T, T_1 \cdots T_p \gamma_{s+1} \cdots \gamma_\omega)$ of C is equal to the smaller of the ranks $r(TT_1 \cdots T_p; 1\Pi_1 \cdots \Pi_1; \Pi_1)$ of A and $r(\gamma_1 \cdots \gamma_s T, \gamma_{s+1} \cdots \gamma_\omega)$ of B ;

2. the rank $r(\gamma_1 \cdots \gamma_s, TT_1 \cdots T_p \gamma_{s+1} \cdots \gamma_\omega)$ of C is equal to the smaller of the ranks $r(TT_1 \cdots T_p; \Pi \cdots \Pi; \Pi)$ of A and $r(\gamma_1 \cdots \gamma_s; T \gamma_{s+1} \cdots \gamma_\omega)$ of B .

For the cases where no partitions are signant we have

THEOREM 36. Let $C = (a_{T_1 \dots T_p T} b_{T \gamma_1 \dots \gamma_\omega})$, T not summed, $A = (a_{T_1 \dots T_p T})$, $B = (b_{T \gamma_1 \dots \gamma_\omega})$. The rank $r(T \gamma_1 \cdots \gamma_\omega T_1 \cdots T_p; \Pi \Pi_1 \cdots \Pi_\omega \Pi_{\omega+1} \cdots \Pi_{\omega+p}; \alpha)$ of C is equal to the smaller of the ranks $r(TT_1 \cdots T_p; \Pi \Pi_{\omega+1} \cdots \Pi_{\omega+p}; \alpha)$ of A and $r(T \gamma_1 \cdots \gamma_\omega; \Pi \Pi_1 \cdots \Pi_\omega; \alpha)$ of B .

All ranks of C are treated in Theorems 34 to 36 except the ranks of C which contain partitions of the type $\rho_1 = TT_1$, $\rho_2 = T_1 T \gamma_1$, $\delta_1 = T_{\epsilon+1} \gamma_{\lambda+1}$, $\delta_2 = T_{\epsilon+2} \gamma_{\lambda+2}, \dots$. Now the $B \times \delta$ device used to treat these ranks under zero-compositions or compositions on T_σ can not be applied to Scott products.

19. An application of ranks of Scott Products and T_σ -composites. We shall call the product $\left(\sum_T a_{TT_1 \dots T_p} b_{T \gamma_1 \dots \gamma_\omega} c_{T \sigma_1 \dots \sigma_p} \right)$ of three matrices $A = (a_{TT_1 \dots T_p})$, $B = (b_{T \gamma_1 \dots \gamma_\omega})$, $C = (c_{T \sigma_1 \dots \sigma_p})$, the triple-composite of these matrices on T . In another paper²⁶ the author has derived the conditions under which certain matrices are factorable into triple-composites. Concerning the ranks of these composites we shall prove

THEOREM 37. Let matrices A, B, C, D be given by $(a_{TT_1 \dots T_p})$, $(b_{T \gamma_1 \dots \gamma_\omega})$, $(c_{T \sigma_1 \dots \sigma_p})$, $(a_{TT_1 \dots T_p} b_{T \gamma_1 \dots \gamma_\omega} c_{T \sigma_1 \dots \sigma_p})$ respectively. Then

1. if $\epsilon, \lambda, \tau > 0$, ϵ, λ are odd, and τ is even, the rank $r(T_1 \cdots T_\epsilon \gamma_1 \cdots \gamma_\lambda \sigma_1 \cdots \sigma_\tau, T_{\epsilon+1} \cdots T_p \gamma_{\lambda+1} \cdots \gamma_\omega \sigma_{\tau+1} \cdots \sigma_p)$ of D is less than or equal to the ranks $r(T_1 \cdots T_\epsilon T, T_{\epsilon+1} \cdots T_p)$ of A , $r(\gamma_1 \cdots \gamma_\lambda T, \gamma_{\lambda+1} \cdots \gamma_\omega)$ of B , and $r(\sigma_1 \cdots \sigma_\tau, T \sigma_{\tau+1} \cdots \sigma_p)$ of C ;

²⁶ Equivalence of pairs of Trilinear Forms; Non-singular case, not yet published.

2. if $\epsilon, \lambda, \tau > 0$, and ϵ, λ, τ are all even, the rank $r(T_1 \cdots T_{\epsilon} \gamma_1 \cdots \gamma_{\lambda} \sigma_1 \cdots \sigma_{\tau}, T_{\epsilon+1} \cdots T_p \gamma_{\lambda+1} \cdots \gamma_{\omega} \sigma_{\tau+1} \cdots \sigma_{\rho})$ of D is less than or equal to the ranks $r(T_1 \cdots T_{\epsilon}, TT_{\epsilon+1} \cdots T_p)$ of A , $r(\gamma_1 \cdots \gamma_{\lambda}, T \gamma_{\lambda+1} \cdots \gamma_{\omega})$ of B , and $r(\sigma_1 \cdots \sigma_{\tau}, T \sigma_{\tau+1} \cdots \sigma_{\rho})$ of C ;

3. if $\epsilon, \tau > 0$, and ϵ, τ are both odd, the rank $r(T_1 \cdots T_{\epsilon} \sigma_1 \cdots \sigma_{\tau}, T_{\epsilon+1} \cdots T_p \gamma_1 \cdots \gamma_{\omega} \sigma_{\tau+1} \cdots \sigma_{\rho})$ of D is less than or equal to the ranks $r(T_1 \cdots T_{\epsilon} T, T_{\epsilon+1} \cdots T_p)$ of A , $r(T \gamma_1 \cdots \gamma_{\omega}; \Pi \cdots \Pi; \Pi)$ of B , and $r(\sigma_1 \cdots \sigma_{\tau} T, \sigma_{\tau+1} \cdots \sigma_{\rho})$ of C , where Π is the smallest of the orders of $T_1, \cdots, T_{\epsilon}, \sigma_1, \cdots, \sigma_{\tau}$;

4. if $\epsilon, \tau > 0$, and ϵ, τ are both even, the rank $r(T_1 \cdots T_{\epsilon} \sigma_1 \cdots \sigma_{\tau}, T_{\epsilon+1} \cdots T_p \gamma_1 \cdots \gamma_{\omega} \sigma_{\tau+1} \cdots \sigma_{\rho})$ of D is less than or equal to the ranks $r(T_1 \cdots T_{\epsilon}, TT_{\epsilon+1} \cdots T_p)$ of A , $r(T \gamma_1 \cdots \gamma_{\omega}; \Pi \cdots \Pi; \Pi)$ of B , and $r(\sigma_1 \cdots \sigma_{\tau}, T \sigma_{\tau+1} \cdots \sigma_{\rho})$ of C , where Π is the smallest of the orders of $T_1, \cdots, T_{\epsilon}, \sigma_1, \cdots, \sigma_{\tau}$;

5. if $\tau > 0$, and τ is even, the rank $r(\sigma_1 \cdots \sigma_{\tau}, T_1 \cdots T_p \gamma_1 \cdots \gamma_{\omega} \sigma_{\tau+1} \cdots \sigma_{\rho})$ of D is less than or equal to the ranks $r(TT_1 \cdots T_p; \Pi \cdots \Pi; \Pi)$ of A , $r(T \gamma_1 \cdots \gamma_{\omega}; \Pi \cdots \Pi; \Pi)$ of B , and $r(\sigma_1 \cdots \sigma_{\tau}, T \sigma_{\tau+1} \cdots \sigma_{\rho})$ of C , where Π is the smallest of the orders of $\sigma_1, \cdots, \sigma_{\tau}$;

6. the rank $r(T_1 \cdots T_p \gamma_1 \cdots \gamma_{\omega} \sigma_1 \cdots \sigma_{\rho}; \mu_1 \cdots \mu_p \xi_1 \cdots \xi_{\omega} \beta_1 \cdots \beta_{\rho}; \alpha)$ of D is less than or equal to the ranks $r(TT_1 \cdots T_p; \alpha \mu_1 \cdots \mu_p; \alpha)$ of A , $r(T \gamma_1 \cdots \gamma_{\omega}; \alpha \xi_1 \cdots \xi_{\omega}; \alpha)$ of B , and $r(T \sigma_1 \cdots \sigma_{\rho}; \alpha \beta_1 \cdots \beta_{\rho}; \alpha)$ of C .

Now $D = E | T | C$, where E is the Scott product $(a_{T T_1} \cdots a_{T T_p} b_{T \gamma_1} \cdots b_{T \gamma_{\omega}})$ of A and B . To prove Cases 1 to 4 of Theorem 37 we obtain inequalities between the ranks of D, E , and C by means of Theorem 22. Furthermore, for Cases 1 and 2 we obtain inequalities between the ranks of E, A , and B by using Theorem 34, and for Cases 3 and 4 by applying Theorem 35.

To treat Case 5 we obtain inequalities between the ranks of D, E , and C by using Theorem 24, and between the ranks of E, A , and B by using Theorem 36.

Finally, for Case 6 we obtain inequalities between the ranks of D, E , and C by means of Theorem 25, and inequalities between the ranks of E, A , and B by Theorem 36.

Theorem 37 covers all of the ranks of D which can be treated by means of the rank theory developed in §§15-17.

20. Relations between the ranks of a matrix. We shall now prove

THEOREM 38. *The rank $r(T_1 \cdots T_{\chi}, T_{\chi+1} \cdots T_{\xi})$, $\chi \geq 2$, of a matrix A is less than or equal to the rank $r(\rho_1 \cdots \rho_{\alpha}, \rho_{\alpha+1} \cdots \rho_{\alpha+\beta})$, $\alpha \geq 2$, of A , where the T 's simply cover the ρ 's, and $\rho_1, \cdots, \rho_{\alpha}$ each contain an odd number of signant T 's, and where $\rho_{\alpha+1}, \cdots, \rho_{\alpha+\beta}$ each contain an even number (≥ 0) of signant T 's.²⁷*

Let the rank of A in the T 's be denoted by $r(T_1 \cdots T_{\nu} T_{\nu+1} \cdots T_{\nu+\epsilon} \cdots T_{\psi} \cdots T_{\chi}, T_{\chi+1} \cdots T_{\chi+\lambda} T_{\chi+\lambda+1} \cdots T_{\chi+\lambda+\omega} \cdots T_{\mu} \cdots T_{\xi})$. Let $|M|_n$ denote an n th-order minor of the $(T_1 \cdots T_{\chi}, T_{\chi+1} \cdots T_{\xi})$ -derivate $(a'_{T_1} \cdots a'_{T_{\chi}} a'_{T_{\chi+1}} \cdots a'_{T_{\xi}})$ of A , and let the $(\rho_1 \cdots \rho_{\alpha}, \rho_{\alpha+1} \cdots \rho_{\alpha+\beta})$ -derivate of A be given by

²⁷ The special case for Hitchcock's ranks is treated by F. L. Hitchcock, Op. Cit., p. 73. He considers relations between the ranks $r(T_1 \cdots T_{p-1}, T_p)$, $r(T_1 \cdots T_p)$, and $r(T_1 \cdots T_r, \alpha)$, $r(T_1 \cdots T_r, \alpha)$, where $\alpha = T_{r+1} \cdots T_p$, $r < p$.

$(a'_{\rho_1 \dots \rho_\alpha \rho'_{\alpha+1} \dots \rho'_{\alpha+\beta}})$. Let $\rho_1 = T_1 \dots T_\nu T_{\chi+1} \dots T_{\chi+\lambda}$, $\rho_2 = T_{\nu+1} \dots T_{\nu+\epsilon}$, $T_{\chi+\lambda+1} \dots T_{\chi+\lambda+\omega}$, $\dots$, $\rho_\alpha = T_\psi \dots T_\chi$, $\dots$, $\rho_{\alpha+\beta} = T_\mu \dots T_\xi$. Let $\rho_{i1}, \rho_{i2}, \dots, \rho_{in}$ be n values of ρ_i , $i = 1, \dots, \alpha$, and $\rho'_{i1}, \rho'_{i2}, \dots, \rho'_{in}$ be n values of ρ'_i for $i = \alpha + 1, \dots, \alpha + \beta$; similarly let $T_{i1}, \dots, T_{in}$ represent n values of T_i for $i = 1, \dots, \chi$, and $T'_{i1}, \dots, T'_{in}$ n values of T'_i for $i = \chi + 1, \dots, \xi$. Let S represent the sum of all terms in the expansion of $|M|_n$ in which $T_{11}, \dots, T_{1n}, T_{21}, \dots, T_{2n}, \dots, T_{\chi 1}, \dots, T_{\chi n}, T'_{\chi+1,1}, \dots, T'_{\chi+1,n}, \dots, T'_{\xi 1}, \dots, T'_{\xi n}$ occur only in the combination $\rho_{j1} = T_{11} \dots T_{\nu 1} T'_{\chi+1,1} \dots T'_{\chi+\lambda,1}$, $\rho_{j2} = T_{12} \dots T_{\nu 2} T'_{\chi+1,2} \dots T'_{\chi+\lambda,2}$, $\dots$, $\rho_{jn} = T_{1n} \dots T_{\nu n} T'_{\chi+1,n} \dots T'_{\chi+\lambda,n}$, $\dots$, $\rho_{\alpha+\beta,1} = T'_{\mu 1} \dots T'_{\xi 1}$, $\dots$, $\rho_{\alpha+\beta,n} = T'_{\mu n} \dots T'_{\xi n}$. Let $\sigma_1, \dots, \sigma_\alpha$ be defined by the relations

$$\begin{aligned} (\rho_{11} \dots \rho_{1n}) \sigma_1 &= [T_{11} \dots T_{1n}] \dots [T_{\nu 1} \dots T_{\nu n}], \\ (\rho_{21} \dots \rho_{2n}) \sigma_2 &= [T_{\nu+1,1} \dots T_{\nu+1,n}] \dots [T_{\nu+\epsilon,1} \dots T_{\nu+\epsilon,n}], \\ &\vdots \\ (\rho_{\alpha 1} \dots \rho_{\alpha n}) \sigma_\alpha &= [T_{\psi 1} \dots T_{\psi n}] \dots [T_{\chi 1} \dots T_{\chi n}]. \end{aligned} \quad (11)$$

Evidently $\sigma_1, \dots, \sigma_\alpha$ are uniquely defined by the choice of $\rho_{11}, \dots, \rho_{1n}, \dots, \rho_{\alpha 1}, \dots, \rho_{\alpha n}$. Now $S = \sigma_1 \dots \sigma_\alpha |M'|_n$ where $|M'|_n$ is the minor of the $(\rho_1 \dots \rho_\alpha, \rho_{\alpha+1} \dots \rho_{\alpha+\beta})$ -derivate of A obtained by letting ρ_i vary over the range $\rho_{i1}, \dots, \rho_{in}$ for $i = 1, \dots, \alpha$, and ρ'_i vary over $\rho'_{i1}, \dots, \rho'_{in}$ for $i = \alpha + 1, \dots, \alpha + \beta$. Furthermore

$$(12) \quad |M|_n = \Sigma \sigma_1 \dots \sigma_\alpha |M'|_n,$$

where²⁸ the summation is over all $\rho_{ij}, i = 1, \dots, \alpha; j = 1, \dots, n$, and $\rho'_{ij}, i = \alpha + 1, \dots, \alpha + \beta; j = 1, \dots, n$ obtained by permuting $T_{11}, \dots, T'_{\xi n}$ in the ρ_{ij} and ρ'_{ij} . This proves the theorem.

By (11) and (12) we also have

THEOREM 39. *The rank $r(T_1 \dots T_\chi, T_{\chi+1} \dots T_\xi), \chi \geq 2$, of a matrix A is less than or equal to the rank $r(\rho_1 \dots \rho_\beta; \Pi \dots \Pi; \Pi)$ of A , where the T 's simply cover the ρ 's, and each of the ρ 's contain an even number of signant T 's, and further Π is the smallest of the orders of $T_1, \dots, T_\chi$.*

For the non-signant ranks of A we have

THEOREM 40. *Let γ_1 be the smallest of a set of positive integers $\mu_1, \dots, \mu_\nu, \alpha; \gamma_2$ be the smallest of a set of positive integers $\mu_{\nu+1}, \dots, \mu_{\nu+\lambda}, \alpha$; etc. Let ρ_1 be a partition of the indices of $A = (a_{T_1 \dots T_\xi})$ which is simply covered by $T_1, \dots, T_\nu$; ρ_2 a partition which is simply covered by $T_{\nu+1}, \dots, T_{\nu+\lambda}$; etc. Then the rank $r(T_1 \dots T_\xi; \mu_1 \dots \mu_\xi; \alpha)$ of A is less than or equal to the rank $r(\rho_1 \dots \rho_\beta; \gamma_1 \dots \gamma_\beta; \alpha)$ of A .*

Let $|M|_n, n \leq \alpha$, denote an n th-order minor of the $(T_1 \dots T_\xi; \mu_1 \dots \mu_\xi)$ -

²⁸ The expansion (12) of $|M|_n$ is given by Rice, Op. Cit., pp. 51-54.

derivate of A . Let this derivate of A be denoted by $(a'_{\tau'_1 \dots \tau'_\xi})$. Let $\rho'_1 = T'_1 \dots T'_{\nu'} \rho'_2 = T'_{\nu'+1} \dots T'_{\nu'+\lambda} \dots$, where $\rho'_1, \dots, \rho'_\beta$ simply cover $T'_1, \dots, T'_\xi$. Let $\rho'_{i1}, \dots, \rho'_{in}$ represent n values of ρ'_i for $i = 1, \dots, \beta$; similarly let $T'_{i1}, \dots, T'_{in}$ represent n values of T'_i for $i = 1, \dots, \xi$. Let S represent the sum of all terms in the expansion of $|M|_n$ in which $T'_{11}, \dots, T'_{1n}, T'_{21}, \dots, T'_{\xi n}$ occur only in the combination $\rho'_{11} = T'_{11} \dots T'_{\nu 1}, \rho'_{12} = T'_{12} \dots T'_{\nu 2}, \dots$. Then $S = |M'|_n$, where $|M'|_n$ is the minor of the $(\rho_1 \dots \rho_\beta; \gamma_1 \dots \gamma_\beta)$ -derivate $(a'_{\rho'_1 \dots \rho'_\beta})$ of A obtained by letting $\rho'_i, i = 1, \dots, \beta$, vary over the range $\rho'_{i1}, \dots, \rho'_{in}$. Further $|M|_n = \Sigma |M'|_n$, where the summation extends over all $\rho'_{ij}, i = 1, \dots, \beta; j = 1, \dots, n$, obtained by permuting $T'_{11}, \dots, T'_{\xi n}$ in the ρ'_{ij} . This completes the proof of the theorem.

21. Invariance of space ranks under non-singular compositions. If the matrix B in Theorems 22, 24, 25 is non-singular on $T_\sigma, \gamma = \gamma_1 \dots \gamma_\omega$, we have the following theorem.

THEOREM 41. *Let A, B, C be given by $(a_{\tau_1 \dots \tau_\sigma}), (b_{\tau_\sigma \gamma}), A | T_\sigma | B$ respectively. Let B be non-singular on T_σ, γ . Let further δ_1, ρ be defined by the relations $\delta_1 = T_{\sigma-1} \gamma, \rho = T_{\sigma-1} T_\sigma$. Then*

1. *the rank $r(T_1 \dots T_r, T_{r+1} \dots T_{\sigma-2} \delta_1)$ of C is equal to the rank $r(T_1 \dots T_r, T_{r+1} \dots T_{\sigma-2} \rho)$ of A ;*
2. *the rank $r(T_1 \dots T_r, T_{r+1} \dots T_{\sigma-1} \gamma)$ of C is equal to the rank $r(T_1 \dots T_r, T_{r+1} \dots T_{\sigma-1} T_\sigma)$ of A ;*
3. *the rank $r(T_1 \dots T_r \delta_1, T_{r+1} \dots T_{\sigma-2})$ of C is equal to the rank $r(T_1 \dots T_r \rho, T_{r+1} \dots T_{\sigma-2})$ of A ;*
4. *the rank $r(T_1 \dots T_r \gamma, T_{r+1} \dots T_{\sigma-1})$ of C is equal to the rank $r(T_1 \dots T_r T_\sigma, T_{r+1} \dots T_{\sigma-1})$ of A ;*
5. *the rank $r(T_1 \dots T_{\sigma-1} \gamma; \mu_1 \dots \mu_{\sigma-1} \alpha; \alpha)$ of C is equal to the rank $r(T_1 \dots T_{\sigma-1} T_\sigma; \mu_1 \dots \mu_{\sigma-1} \alpha; \alpha)$ of A ;*
6. *the rank $r(T_1 \dots T_{\sigma-2} \delta_1; \mu_1 \dots \mu_{\sigma-2} \alpha; \alpha)$ of C is equal to the rank $r(T_1 \dots T_{\sigma-2} \rho; \mu_1 \dots \mu_{\sigma-2} \alpha; \alpha)$ of A .*

Let $\delta = (\delta_{\tau_{\sigma-1} \tau'_{\sigma-1}})$ be an identity matrix on $T_{\sigma-1}, T'_{\sigma-1}$, and let $\delta'_1 = T'_{\sigma-1} \gamma$. The theorem is proved by means of the relations $C = A | T_\sigma | B, A = C | \gamma | B_{T_\sigma}^{-1} \gamma$, Theorems 22, 24, 25, and the ranks of $C = A | \rho | B \times \delta, A = C | \delta'_1 | (B \times \delta)^{-1} \delta'_1$.

22. Space ranks of $A \times B$ where B is non-singular. If in Theorems 27-33 B is non-singular on γ_1, γ_2 , we have the following theorem.

THEOREM 42. *Let $A = (a_{\tau_1 \dots \tau_p}), B = (b_{\gamma_1 \gamma_2}), C = A \times B$. Let partitions $\gamma, \delta_0, \delta_1, \delta_2, \rho, \rho_1$ be defined by the relations $\gamma = \gamma_1 \gamma_2, \delta_0 = T_{s+1} \gamma_1 \gamma_2, \delta_1 = T_{s+1} \gamma_1, \delta_2 = T_{s+2} \gamma_2, \rho = T_{s+1} T_{s+2}$. Assume that B is non-singular on γ_1, γ_2 . Let Π be the order of γ_1, Π_1 the smallest of the orders of δ_1, δ_2 , and let Π_2 be the smallest of the orders of $T_1, \dots, T_s$. Then the ranks of A and C which occur in corresponding position in the following table are equal.*

$T_{s+2} \cdots T_p; \mu_1 \cdots \mu_s \alpha \mu_{s+2} \cdots \mu_p; \alpha)$, $r(\rho T_1 \cdots T_s T_{s+3} \cdots T_p; \alpha \mu_1 \cdots \mu_s \mu_{s+3} \cdots \mu_p; \alpha)$ of A . If $s \geq 2$, $\Pi \geq \Pi_2$, the ranks $r(T_1 \cdots T_s \gamma_1 \gamma'_1, T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_s \delta_1 \gamma'_1, T_{s+2} \cdots T_p)$ of $A \times \delta$ are respectively equal to the ranks $r(T_1 \cdots T_s, T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_s, T_{s+1} \cdots T_p)$ of A . Finally, if $s \geq 2$, the rank $r(T_1 \cdots T_s \delta_1 \delta'_2, T_{s+3} \cdots T_p)$ of $A \times \delta$ is equal to the rank $r(T_1 \cdots T_s, T_{s+3} \cdots T_p)$ of A .

By the above relations between the ranks of A and $A \times \delta$, and Theorems 22, 24, 25, 27-33, 45, 46, Theorem 42 is proved.

From Theorem 42 several equalities may be obtained between the ranks of C .

23. Space ranks of $C = (a_{T_1 \cdots T_p T} b_{T\gamma})$, T not summed, where $B = (b_{T\gamma})$ is non-singular on T, γ . If we let $B_{T\gamma}^{-1} = (b^{T'\gamma})$, then

$$C | \gamma | B_{T\gamma}^{-1} = (a_{T_1 \cdots T_p T} b_{T\gamma} b^{T'\gamma}) = (a_{T_1 \cdots T_p T} \delta_{TT'}),$$

where $\delta = (\delta_{TT'})$ is an identity matrix on T, T' . We shall now prove

THEOREM 43. *Let $A = (a_{T_1 \cdots T_p T})$, $B = (b_{T\gamma})$, $C = (a_{T_1 \cdots T_p T} b_{T\gamma})$. If B is non-singular on T, γ , and we let Π denote the order of γ , then the ranks $r(T_1 \cdots T_s, \gamma T T_{s+1} \cdots T_p)$, $r(\gamma T, T_1 \cdots T_p)$, $r(T T_1 \cdots T_p \gamma; \mu \mu_1 \cdots \mu_p \beta; \alpha)$, $\beta \geq \mu$, $r(T_1 \cdots T_s \gamma, T T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_s T, \gamma T_{s+1} \cdots T_p)$ of C are equal respectively to the ranks $r(T_1 \cdots T_s, T T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_p T; \Pi \cdots \Pi 1; \Pi)$, $r(T T_1 \cdots T_p; \mu \mu_1 \cdots \mu_p; \alpha)$, $r(T_1 \cdots T_s T, T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_s T, T_{s+1} \cdots T_p)$ of A .*

It is easily shown that the ranks $r(T_1 \cdots T_s, T T' T_{s+1} \cdots T_p)$, $r(T' T, T_1 \cdots T_p)$, $r(T T_1 \cdots T_p T'; \mu \mu_1 \cdots \mu_p \beta; \alpha)$, $\beta \geq \mu$, $r(T_1 \cdots T_s T', T_{s+1} \cdots T_p T)$, $r(T_1 \cdots T_s T, T' T_{s+1} \cdots T_p)$ of $(a_{T_1 \cdots T_p T} \delta_{TT'})$ are respectively equal to the ranks $r(T_1 \cdots T_s, T T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_p T; \Pi \cdots \Pi 1; \Pi)$, $r(T_1 \cdots T_p T; \mu_1 \cdots \mu_p \mu; \alpha)$, $r(T_1 \cdots T_s T, T_{s+1} \cdots T_p)$, $r(T_1 \cdots T_s T, T_{s+1} \cdots T_p)$ of A . By these relations and Theorems 35 and 36, the theorem is proved.

In general the rank $r(T_1 \cdots T_s T T', T_{s+1} \cdots T_p)$ of $(a_{T_1 \cdots T_p T} \delta_{TT'})$ is not greater than or equal to the rank $r(T_1 \cdots T_s, T T' T_{s+1} \cdots T_p)$ of A , whence the rank $r(T_1 \cdots T_s T \gamma, T_{s+1} \cdots T_p)$ of C cannot be treated by our method.

24. Remarks on §§21-23. It can now be shown that the restriction that the matrix B be non-singular on two partitions, where these partitions simply cover the total partition of the indices of B , can be *weakened* to non-singularity on T_σ in §21, non-singularity on γ_2 in §22, and non-singularity on T in §23. The theorems of these paragraphs can be restated in terms of these non-singularity properties.

Adopting the term *invariance* in the sense of the equality of ranks in §§21 and 22 we can state

THEOREM 44. *The signant space ranks of a matrix $A = (a_{T_1 \cdots T_p})$ are invariant under compositions on T_i with $B_i = (b_{T_i \gamma_i})$, $i = 1, \cdots, p$, where B_i is non-singular on T_i . They are also invariant under zero composition with a matrix $B = (b_{T\gamma})$, where B is non-singular on T .*

In view of the Lemma and the equivalence of a set of linear transformations to a single composition we have the

COROLLARY. *The signant ranks of a matrix A are invariant under non-singular linear transformations.*

Such invariant properties have also been established wherever possible for a large class of signant ranks under Scott multiplication, and for a large class of non-signant ranks under the different possible compositions and Scott multiplication.

25. Ranks of $A \times B \times C \times \dots$. We shall prove the following generalization of Theorem 8, keeping the conventions concerning vacuous partitions adopted in §10.

THEOREM 45. *Let $\alpha = A_1 \times A_2 \times A_3 \dots$ where $A_1, A_2, A_3, \dots$ are matrices. Let $\rho_1, \dots, \rho_{s+r+t+w}$ completely cover the total partition of the indices of α , and let $\rho_1 = \rho_1 \sigma_1, \dots, \rho_s = \rho_s \sigma_s, \rho_{s+r+1} = \rho_{s+r+1} \sigma_{s+r+1}, \dots, \rho_{s+r+t} = \rho_{s+r+t} \sigma_{s+r+t}$ where $\sigma_1, \dots, \sigma_s, \sigma_{s+r+1}, \dots, \sigma_{s+r+t}$ are total partitions of the factor matrices $A_1, \dots, A_s, A_{s+r+1}, \dots, A_{s+r+t}$ respectively of α . Then the rank $r(\rho_1 \dots \rho_s \rho_{s+1} \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t} \rho_{s+r+t+1} \dots \rho_{s+r+t+w})$, $s+r > 0$, of α is equal to the rank $r(\rho_1' \dots \rho_s' \rho_{s+1} \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t} \rho_{s+r+t+1} \dots \rho_{s+r+t+w})$ of α' , where α' is obtained from α by deleting the matrices $A_1, \dots, A_s, A_{s+r+1}, \dots, A_{s+r+t}$.*

If $A_1 = (a_{\sigma_1})$, then A_1 is equivalent under composition with a matrix C non-singular on σ_1 to a matrix $A_1' = (a_{\sigma_1}')$, where $a_{\sigma_1}'^* = 1$ if $\sigma_1^* = 1$, $a_{\sigma_1}'^* = 0$ if $\sigma_1^* \neq 1$. By Theorem 41 the rank $r(\rho_1 \dots \rho_s \rho_{s+1} \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t} \rho_{s+r+t+1} \dots \rho_{s+r+t+w})$ of α is equal to the rank $r(\rho_1' \rho_2 \dots \rho_s \rho_{s+1} \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t} \rho_{s+r+t+1} \dots \rho_{s+r+t+w})$ of $A_1' \times A_2 \times A_3 \dots$, whence Theorem 45 is proved.

COROLLARY 1. *If α is not a zero matrix, and if one of the partitions $\rho_1, \dots, \rho_s$ of Theorem 45 is the total partition of one of the factor matrices $A_1, A_2, A_3, \dots$, then the rank $r(\rho_1 \dots \rho_s \rho_{s+1} \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t+w})$, $s+r > 0$, of α is unity.*

COROLLARY 2. *If the partition ρ_{s+r+t} of Theorem 45 is the total partition of the factor matrix A_{s+r+t} , then the rank $r(\rho_1 \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t} \rho_{s+r+t+1} \dots \rho_{s+r+t+w})$ of α is equal to the rank $r(\rho_1 \dots \rho_{s+r}, \rho_{s+r+1} \dots \rho_{s+r+t-1} \rho_{s+r+t+1} \dots \rho_{s+r+t+w})$ of $\alpha' = A \times \dots \times A_{s+r+t-1} \times A_{s+r+t+1} \times \dots$.*

By a proof similar to that used for Theorem 45 we are able to prove for non-signant ranks

THEOREM 46. *Let $\alpha = A_1 \times A_2 \times A_3 \dots$, where $A_1, A_2, A_3, \dots$, are matrices. The rank $r(\rho_1 \dots \rho_s \rho_{s+1} \dots \rho_{s+r}; \mu_1 \dots \mu_s \mu_{s+1} \dots \mu_{s+r}; \beta)$, $\beta = \mu_1 = \mu_2 = \dots = \mu_s$, of α is equal to the rank $r(\rho_1' \dots \rho_s' \rho_{s+1} \dots \rho_{s+r}; \mu_1 \dots \mu_s \mu_{s+1} \dots \mu_{s+r}; \beta)$ of α' , where $\rho_1 = \rho_1' \sigma_1, \dots, \rho_s = \rho_s' \sigma_s$, and $\sigma_1, \dots, \sigma_s$ are total partitions of the factor matrices $A_1, \dots, A_s$ of α ; and further α' is obtained by deleting $A_1, \dots, A_s$ from α .*

We have the

COROLLARY. *If in Theorem 46 the partition ρ_1 is the total partition of the factor matrix A_1 of α , the rank $r(\rho_1 \cdots \rho_s \rho_{s+1} \cdots \rho_{s+r}; \mu_1 \cdots \mu_s \mu_{s+1} \cdots \mu_{s+r}; \beta)$, $s + r \geq 2$, of α is equal to the rank $r(\rho_2 \rho_3 \cdots \rho_s \rho_{s+1} \cdots \rho_{s+r}; \mu_2 \mu_3 \cdots \mu_s \mu_{s+1} \cdots \mu_{s+r}; \beta)$ of $\alpha' = A_2 \times A_3 \times \cdots$.*

26. Application of the Lemma to rank properties under compositions. Although as mentioned in §16 the ranks of a general $B \times \delta$ are not readily expressible in terms of the ranks of B , by means of the Lemma certain 2-way ranks of $B \times \delta$ can be given in terms of the ranks of B . For the ranks of composites we have

THEOREM 47. *Let $A = (a_{T_1 \cdots T_\omega \cdots T_{\omega+p} \cdots T_\sigma})$, $B = (b_{T_\sigma \gamma_1 \cdots \gamma_\omega})$, $\omega > 0$, $C = A | T_\sigma | B$. Let $\delta_1 = T_1 \gamma_1, \cdots, \delta_\omega = T_\omega \gamma_\omega$. Let also $\delta = (\delta_{\gamma_1 \gamma'_1}) \times (\delta_{\gamma_2 \gamma'_2}) \times \cdots \times (\delta_{\gamma_{\omega-1} \gamma'_{\omega-1}})$, where $(\delta_{\gamma_i \gamma'_i})$, $i = 1, \cdots, \omega - 1$ is an identity matrix on γ_i, γ'_i . Let $\delta^* = (\delta_{T_\omega T'_\omega})$ be an identity matrix on T_ω, T'_ω , and let $\delta'_1 = T_1 \gamma'_1, \cdots, \delta'_{\omega-1} = T_{\omega-1} \gamma'_{\omega-1}$, $\delta'_\omega = T'_\omega \gamma_\omega$. Further let Π be the order of T_ω , and let $\rho = T_\sigma \gamma_1 \cdots \gamma_{\omega-1} T_\omega$, $\rho_1 = T_\sigma \gamma_1 \cdots \gamma_{\omega-1}$. Assume that $\sigma > \omega$. Then the rank $r(\delta_1 \cdots \delta_s T_{\omega+1} \cdots T_{\omega+p} \delta_\omega, \delta_{s+1} \cdots \delta_{\omega-1} T_{\omega+p+1} \cdots T_{\sigma-1})$ of C is less than or equal to the rank $r(\delta'_1 \cdots \delta'_s T_{\omega+1} \cdots T_{\omega+p} \rho, \delta'_{s+1} \cdots \delta'_{\omega-1} T_{\omega+p+1} \cdots T_{\sigma-1})$ of $A \times \delta$ and Πr , where r is the rank $r(\rho_1 \gamma_\omega)$ of B .*

The rank $r(\delta_1 \cdots \delta_s T_{\omega+1} \cdots T_{\omega+p} \delta_\omega, \delta_{s+1} \cdots \delta_{\omega-1} T_{\omega+p+1} \cdots T_{\sigma-1})$ of C is equal to the rank $r(\delta'_1 \cdots \delta'_s T_{\omega+1} \cdots T_{\omega+p} \delta'_\omega, \delta'_{s+1} \cdots \delta'_{\omega-1} T_{\omega+p+1} \cdots T_{\sigma-1})$ of C_1 , where

$$C_1 = (A \times \delta) | \rho | (B \times \delta^*).$$

The theorem now follows by means of Theorem 22 and the Lemma.

For the ranks of zero-composites we have

THEOREM 48. *Let $A = (a_{T_1 \cdots T_\omega \cdots T_{\omega+p} \cdots T_\sigma})$, $B = (b_{T_\sigma \gamma_1 \cdots \gamma_\omega})$, $C = A \times B$. Let $\delta_1 = T_1 \gamma_1, \cdots, \delta_\omega = T_\omega \gamma_\omega$, and let $\delta = (\delta_{\gamma_1 \gamma'_1}) \times \cdots \times (\delta_{\gamma_{\omega-1} \gamma'_{\omega-1}})$, where $(\delta_{\gamma_i \gamma'_i})$, $i = 1, \cdots, \omega - 1$ is an identity matrix on γ_i, γ'_i . Let $\delta^* = (\delta_{T_\omega T'_\omega})$ be an identity matrix on T_ω, T'_ω . Let also $\delta'_1 = T_1 \gamma'_1, \cdots, \delta'_{\omega-1} = T_{\omega-1} \gamma'_{\omega-1}$, $\delta'_\omega = T'_\omega \gamma_\omega$. Let Π be the order of T_ω , and let $\rho = \gamma_1 \cdots \gamma_{\omega-1} T_\omega$, $\rho_1 = \gamma_1 \cdots \gamma_{\omega-1}$. Then the rank $r(\delta_1 \cdots \delta_s T_{\omega+1} \cdots T_{\omega+p} \delta_\omega, \delta_{s+1} \cdots \delta_{\omega-1} T_{\omega+p+1} \cdots T_{\sigma-1})$ of C is less than or equal to the rank $r(\delta'_1 \cdots \delta'_s T_{\omega+1} \cdots T_{\omega+p} \rho, T_{\omega+p+1} \cdots T_{\sigma-1})$ of $A \times \delta$ and Πr , where, if $\omega > 1$, r is the rank $r(\rho_1 \gamma_\omega)$ of B , while if $\omega = 1$, $r = 1$ or 0 according as $B \neq 0$ or $B = 0$.*

To prove this theorem we construct a matrix C_1 as in the proof of Theorem 47; then apply the Lemma and Theorem 22.

The ranks covered in Theorems 47 and 48 are the most general ranks which can be treated by applying the Lemma to matrices of the type $B \times \delta$.

27. Space determinants as relative invariants. Theorem 20 and the Lemma imply

THEOREM 49. *If a matrix $A = (a_{T_1 \cdots T_p})$ possesses a determinant $|a_{T_1 \cdots T_p}|$*

signant on $T_1, T_2, \dots, T_r$, then the corresponding determinant of C obtained from A under composition with $B_{ij} = (b_{T_{i1} T_{i2}}^j)$ on $T_{i1}^j, i = 1, 2, \dots, n_j$, simply covering $T_j, j = 1, \dots, r$, is equal to

$$a_{T_1 \dots T_p} \mid \mid b_{T_{11} T_{12}}^1 \mid \mid \Pi_2^1 \dots \Pi_{n_1}^1 \dots \mid \mid b_{T_{n_1 1} T_{n_1 2}}^1 \mid \mid \Pi_1^1 \dots \Pi_{n_1-1}^1 \mid \mid \\ \mid \mid b_{T_{11} T_{12}}^2 \mid \mid \Pi_2^2 \dots \Pi_{n_2}^2 \dots \mid \mid b_{T_{n_r 1} T_{n_r 2}}^r \mid \mid \Pi_1^r \dots \Pi_{n_r-1}^r,$$

where Π_i^j is the order of T_{i1}^j .

The notations of the theorem imply that the matrices $(b_{T_{i1} T_{i2}}^j)$ are square as displayed.

Relative invariant properties under compositions do not hold for determinants non-signant on all indices, or for compositions on non-signant indices of determinants signant on two or more indices.

28. Space ranks of identity matrices. We shall prove

THEOREM 50. Let $\delta = (\delta_{i_1 \dots i_r j_1 \dots j_r}) = (\delta_{i_1 j_1}) \times \dots \times (\delta_{i_r j_r})$, where $(\delta_{i_s j_s})$ is an identity matrix on i_s, j_s for $s = 1, \dots, r$. Let also $\delta = (\delta_{\rho_1 \dots \rho_w \rho'_1 \dots \rho'_w}) = (\delta_{\rho_1 \rho'_1}) \times \dots \times (\delta_{\rho_w \rho'_w})$, where the ρ 's are simply covered by $i_1, \dots, i_r, j_1, \dots, j_r$, and $(\delta_{\rho_s \rho'_s})$ is an identity matrix on ρ_s, ρ'_s for $s = 1, \dots, w$. Let $\sigma_1, \dots, \sigma_{m+l}$ be simply covered by the j indices, and let $\rho = i_1 \dots i_r$. Let also $i_1, \dots, i_r, j_1, \dots, j_r$ simply cover partitions $\beta_1, \dots, \beta_p$. Then

1. the rank $r(\beta_1 \dots \beta_t, \beta_{t+1} \dots \beta_p)$, $t \geq 2$, of δ is unity if one of the signant partitions $\beta_1, \dots, \beta_t$ contains only subsets of the type $i_a j_a$;

2. the rank $r(\rho_1 \rho'_1 \rho_2 \rho'_2 \dots \rho_j \rho'_j \rho_{j+1} \dots \rho'_j \rho'_{j+1} \dots \rho_w \rho'_w)$ of δ is equal to one if partitions $\rho_j, \rho_{j+1}, \dots$, exist among the signant partitions for which $\rho'_j, \rho'_{j+1}, \dots$, occur among the non-signant partitions; otherwise this rank is equal to the smallest of the orders of the partitions $\rho_1, \dots, \rho_w$;

3. the rank $r(\rho \sigma_1 \dots \sigma_m, \sigma_{m+1} \dots \sigma_{m+l})$ of δ is equal to the smallest of the orders of $\sigma_1, \dots, \sigma_{m+l}$; also the rank $r(\sigma_1 \dots \sigma_m, \sigma_{m+1} \dots \sigma_{m+l} \rho)$ of δ is equal to the smallest of the orders of the partitions $\sigma_1, \dots, \sigma_{m+l}$;

4. the rank $r(\rho_1 \rho'_1 \dots \rho_w \rho'_w; \mu_1 \mu'_1 \dots \mu_w \mu'_w; \alpha)$ of δ is equal to α .

Part 1 of the theorem follows from Cor. 1 of Theorem 45.

To prove part 2 consider a q th-order ($q \geq 2$) minor of $(\delta_{\rho_1 \dots \rho_w})$ given by the determinant $d = \mid \delta_{\rho_1 \rho'_1 \rho_2 \rho'_2} \dots \rho_j \rho'_j \rho_{j+1} \dots \rho_l, \rho'_j \rho'_{j+1} \dots \rho_w \rho'_w \mid$, where the partitions to the left of the comma in d are signant and ρ_i, ρ'_i vary over the same ranges. Let $\rho_{i_1}, \dots, \rho_{i_q}$ denote q values of $\rho_i, i = 1, \dots, w$; similarly for ρ'_i . The sign of $\rho_1, \rho'_1, \rho_2, \rho'_2, \dots$, does not affect the sign of the terms in the expansion of d since $[\rho_{11} \dots \rho_{1q}] [\rho'_{11} \dots \rho'_{1q}] = [\rho_{21} \dots \rho_{2q}] [\rho'_{21} \dots \rho'_{2q}] = \dots = 1$. Hence the expansion of d is a sum of positive terms of the type d_1 , where $d_1 = \mid d_{\rho_j \rho'_j \rho_{j+1}} \dots \mid$, and $d_{\rho_j \rho'_j \rho_{j+1}} = 1$ for all values of $\rho_j, \rho_{j+1}, \dots$; further all partitions exhibited in d_1 are signant, whence $d_1 = 0$.

If there are no partitions of the type $\rho_j, \rho_{j+1}, \dots$, in d , then all terms in the expansion of d are positive and $d \neq 0$.

To prove part 3 let Π be the smallest of the orders of $\sigma_1, \dots, \sigma_{m+l}$. Let a minor of $(\delta_{\rho\sigma_1} \dots \sigma_{m+l})$ be given by the determinant $|E| = |\delta_{\rho\sigma_1} \dots \sigma_{m+l}|$, where σ_i varies over a range $\sigma_{i1}, \dots, \sigma_{i\Pi}$ for $i = 1, \dots, m+l$, and ρ varies over the range $\sigma_{11}\sigma_{21} \dots \sigma_{m+l,1}, \dots, \sigma_{1\Pi}\sigma_{2\Pi} \dots \sigma_{m+l,\Pi}$. For every non-vanishing term in the expansion of $|E|$ the following relations hold:

$$(13) \quad [\rho_{11} \dots \rho_{1q}] = [\sigma_{11} \dots \sigma_{1q}] = \dots = [\sigma_{m1} \dots \sigma_{mq}].$$

Since there are an even number of partitions signant (13) implies that $|E| \neq 0$. We can similarly show that the determinant $|E|$ expanded with $\rho, \sigma_{m+1}, \dots, \sigma_{m+l}$ non-signant and $\sigma_1, \dots, \sigma_{m-1}, \sigma_m$ signant is not zero.

Finally consider a minor of the $(\rho_1\rho_1' \dots \rho_w\rho_w'; \mu_1\mu_1' \dots \mu_w\mu_w')$ -derivate of δ obtained by letting ρ_j, ρ_j' have the same range for $j = 1, \dots, w$, where the ρ 's have α values. Since all terms in the expansion of this minor are positive, it is not zero. This proves part 4 of the theorem.

29. Relation between the 3-way and 2-way ranks of a matrix. The proof of the following theorem is a modification of that given by Hitchcock²⁹ for triadics.

THEOREM 51. *Let partitions $\alpha, \beta, \rho, T, \sigma$ be related by the identities $\alpha = T\sigma, \beta = \rho\sigma$. If the rank $r(\rho T, \sigma)$ of the matrix $(a_{\rho T \sigma})$ is unity, then one of the ranks $r(\rho\alpha), r(T\beta)$ of $(a_{\rho T \sigma})$ is unity.*

By assumption, if ρ_1, ρ_2 denote different values of ρ, T_1, T_2 different values of T , and σ_1, σ_2 different values of σ , then

$$(14) \quad \sum_{\begin{pmatrix} \rho_1 \rho_2 \\ \tau_1 \tau_2 \end{pmatrix}} [\rho_1' \rho_2'] [T_1 T_2] a_{\rho_1 \sigma_1 T_1} a_{\rho_2 \sigma_2 T_2} = 0$$

for every choice of $\rho_1, \rho_2, T_1, T_2, \sigma_1, \sigma_2$. If $\sigma_1 = \sigma_2$, (14) implies that $a_{\rho T \sigma} = \alpha_{\rho}^{\sigma} \beta_T^{\sigma}$. Substituting in (14) with $\sigma_1 \neq \sigma_2$, we get

$$(15) \quad \alpha_{\rho_1}^{\sigma_1} \beta_{T_1}^{\sigma_1} \alpha_{\rho_2}^{\sigma_2} \beta_{T_2}^{\sigma_2} - \alpha_{\rho_2}^{\sigma_1} \beta_{T_1}^{\sigma_1} \alpha_{\rho_1}^{\sigma_2} \beta_{T_2}^{\sigma_2} + \alpha_{\rho_2}^{\sigma_1} \beta_{T_2}^{\sigma_1} \alpha_{\rho_1}^{\sigma_2} \beta_{T_1}^{\sigma_2} - \alpha_{\rho_1}^{\sigma_1} \beta_{T_2}^{\sigma_1} \alpha_{\rho_2}^{\sigma_2} \beta_{T_1}^{\sigma_2} = 0,$$

whence we have

$$(16) \quad (\alpha_{\rho_1}^{\sigma_1} \alpha_{\rho_2}^{\sigma_2} - \alpha_{\rho_2}^{\sigma_1} \alpha_{\rho_1}^{\sigma_2}) (\beta_{T_1}^{\sigma_1} \beta_{T_2}^{\sigma_2} - \beta_{T_2}^{\sigma_1} \beta_{T_1}^{\sigma_2}) = 0.$$

Now the second order minors of $(a_{\rho\alpha})$ are given by

$$\beta_{T_1}^{\sigma_1} \beta_{T_2}^{\sigma_2} \begin{vmatrix} \alpha_{\rho_1}^{\sigma_1} & \alpha_{\rho_1}^{\sigma_2} \\ \alpha_{\rho_2}^{\sigma_1} & \alpha_{\rho_2}^{\sigma_2} \end{vmatrix},$$

²⁹ F. L. Hitchcock, Op. Cit., pp. 70 ff.

The theorem implies the condition $|\sigma U + \tau W| \neq 0$, which occurs in the author's paper "On canonical binary trilinear forms," Bulletin of the American Mathematical Society, vol. 38 (1932), p. 387.

while the corresponding minors of $(a_{T\beta})$ are given by

$$\alpha_{\rho 1}^{\sigma 1} \alpha_{\rho 2}^{\sigma 2} \begin{vmatrix} \beta_{T1}^{\sigma 1} & \beta_{T1}^{\sigma 2} \\ \beta_{T2}^{\sigma 1} & \beta_{T2}^{\sigma 2} \end{vmatrix}.$$

Assume that the rank $r(\rho\alpha) > 1$, whence some minor of $(\alpha_{\rho\alpha})$ is not zero. Let this minor be given by $\beta_{i_1 1}^{\sigma 1} \beta_{i_2 2}^{\sigma 2} (\alpha_{r1}^{\sigma 1} \alpha_{r2}^{\sigma 2} - \alpha_{r2}^{\sigma 1} \alpha_{r1}^{\sigma 2})$. Now at least one of the products $\alpha_{r1t_1s_1} a_{r2t_2s_2}$, $a_{r1t_2s_2} a_{r2t_1s_1}$ is not zero. In our proof we shall assume that $a_{r1t_1s_1} a_{r2t_2s_2} \neq 0$.

We wish to prove that $r(T\beta) = 1$. It is evident that it is sufficient to prove only that all second order minors of $(a_{T\beta})$ which contain $a_{r1t_1s_1}$ are zero, i.e. that $\alpha_{r1}^{\sigma 1} \alpha_{\rho}^{\sigma} (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma}) = 0$ for all ρ, T, σ ; or what is the same, that

$$(17) \quad \alpha_{\rho}^{\sigma} (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma}) = 0$$

for all ρ, T, σ .

If $\alpha_{\rho}^{\sigma 3} = 0$ for all ρ , or if $\beta_T^{\sigma 3} = 0$ for all T , (17) is satisfied. We hence assume that $a_{r3t_3s_3} = \alpha_{r3}^{\sigma 3} \beta_{i_3 3}^{\sigma 3} \neq 0$. By (16) we have

$$(18) \quad (\alpha_{r1}^{\sigma 1} \alpha_{r2}^{\sigma 2} - \alpha_{r2}^{\sigma 1} \alpha_{r1}^{\sigma 2}) (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 2} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 2}) = 0,$$

$$(19) \quad (\alpha_{r1}^{\sigma 1} \alpha_{r3}^{\sigma 3} - \alpha_{r3}^{\sigma 1} \alpha_{r1}^{\sigma 3}) (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 3} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 3}) = 0,$$

$$(20) \quad (\alpha_{r1}^{\sigma 2} \alpha_{r2}^{\sigma 3} - \alpha_{r2}^{\sigma 2} \alpha_{r1}^{\sigma 3}) (\beta_{i_1 1}^{\sigma 2} \beta_T^{\sigma 3} - \beta_T^{\sigma 2} \beta_{i_1 1}^{\sigma 3}) = 0.$$

By (18) we have

$$(21) \quad (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 2} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 2}) = 0.$$

If $\beta_{i_1 1}^{\sigma 2} \neq 0$, (20) and (21) give

$$(22) \quad (\alpha_{r1}^{\sigma 2} \alpha_{r2}^{\sigma 3} - \alpha_{r2}^{\sigma 2} \alpha_{r1}^{\sigma 3}) (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 3} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 3}) = 0.$$

If also $\alpha_{r2}^{\sigma 3} \neq 0$, and $(\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 3} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 3}) \neq 0$, then the equation

$$(23) \quad (\alpha_{r1}^{\sigma 1} \alpha_{r2}^{\sigma 3} - \alpha_{r2}^{\sigma 1} \alpha_{r1}^{\sigma 3}) = (\alpha_{r1}^{\sigma 2} \alpha_{r2}^{\sigma 3} - \alpha_{r2}^{\sigma 2} \alpha_{r1}^{\sigma 3}) = 0,$$

which follows from (19) and (22), contradicts the assumption that $(\alpha_{r1}^{\sigma 1} \alpha_{r2}^{\sigma 2} - \alpha_{r2}^{\sigma 1} \alpha_{r1}^{\sigma 2}) \neq 0$. Hence $\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 3} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 3} = 0$.

If $\beta_{i_1 1}^{\sigma 2} \neq 0$, $\alpha_{r2}^{\sigma 3} = 0$, and further $(\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 3} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 3}) \neq 0$, by (22) $\alpha_{r1}^{\sigma 3} = 0$. Now from (16) we have

$$(24) \quad (\alpha_{r1}^{\sigma 1} \alpha_{\rho}^{\sigma 3} - \alpha_{\rho}^{\sigma 1} \alpha_{r1}^{\sigma 3}) (\beta_{i_1 1}^{\sigma 1} \beta_T^{\sigma 3} - \beta_T^{\sigma 1} \beta_{i_1 1}^{\sigma 3}) = 0$$

for all ρ . This implies that $\alpha_{\rho}^{\sigma 3} = 0$ for all ρ , which contradicts the assumption that $a_{r3t_3s_3} \neq 0$.

If finally $\beta_{i_1 1}^{\sigma 2} = 0$, by (21) $\beta_T^{\sigma 2} = 0$ for all T , which implies that $\beta_{i_2 2}^{\sigma 2} = 0$, giving a contradiction.

30. Compound matrices. Let $A = (a_{T_1 \dots T_\sigma})$, where $T_1, \dots, T_\sigma$ are each of order n . Using double subscripts on partitions in the sense of Section 14, we define a form

$$F(A) = \sum_{\substack{(T_{11} \dots T_{1n}) \\ \vdots \\ (T_{\sigma-1,1} \dots T_{\sigma-1,n})}} [T_{11} \dots T_{1n}] \dots [T_{\sigma-1,1} \dots T_{\sigma-1,n}] a_{T_{11} \dots T_{\sigma 1}} \dots \\ a_{T_{1q} \dots T_{\sigma q}} z_{T_{\sigma 1}}^{(1)} \dots z_{T_{\sigma q}}^{(q)} y_{T_{1,q+1}}^{(q+1)} v_{T_{2,q+1}}^{(q+1)} \dots \\ u_{T_{\sigma-1,q+1}}^{(q+1)} \dots y_{T_{1n}}^{(n)} v_{T_{2,n}}^{(n)} \dots u_{T_{\sigma-1,n}}^{(n)}.$$

The elements of the matrix A_1 of $F(A)$ are the signed determinants of the q th order minors of the $(T_1 \dots T_{\sigma-1}, T_\sigma)$ -derivate of A . A_1 is called the $(T_1 \dots T_{\sigma-1}, T_\sigma)$ -compound³⁰ of A . Let further

$$F(A | T_1 | B) = \sum_{\substack{(\rho_{11} \dots \rho_{1n}) \\ (T_{21} \dots T_{2n}) \\ \vdots \\ (T_{\sigma-1,1} \dots T_{\sigma-1,n})}} [\rho_{11} \dots \rho_{1n}] [T_{21} \dots T_{2n}] \dots [T_{\sigma-1,1} \dots T_{\sigma-1,n}] \\ a_{T_{11}} \dots a_{T_{\sigma 1}} \dots a_{T_{1q}} \dots a_{T_{\sigma q}} b_{T_{11}\rho_{11}} \dots b_{T_{1q}\rho_{1q}} \\ z_{T_{\sigma 1}}^{(1)} \dots z_{T_{\sigma q}}^{(q)} w_{\rho_{1,q+1}}^{(q+1)} \dots w_{\rho_{1n}}^{(n)} v_{T_{2,q+1}}^{(q+1)} \\ \dots u_{T_{\sigma-1,q+1}}^{(q+1)} v_{T_{2n}}^{(n)} \dots u_{T_{\sigma-1,n}}^{(n)}.$$

If $q < n$, $F(A | T_1 | B)$ can be obtained except for a constant factor ($\neq 0$) from $F(A)$ by the following composition

$$(25) \quad y_{T_{1,q+1}}^{(q+1)} \dots y_{T_{1n}}^{(n)} = \sum_{\substack{(T_{11} \dots T_{1q}) \\ (\rho_{11} \dots \rho_{1q})}} [T_{11} \dots T_{1q}] [\rho_{11} \dots \rho_{1q}] b_{T_{11}\rho_{11}} \dots b_{T_{1q}\rho_{1q}} \\ w_{\rho_{1,q+1}}^{(q+1)} \dots w_{\rho_{1n}}^{(n)}.$$

The coefficients $\sum_{\substack{(T_{11} \dots T_{1q}) \\ (\rho_{11} \dots \rho_{1q})}} [T_{11} \dots T_{1q}] [\rho_{11} \dots \rho_{1q}] b_{T_{11}\rho_{11}} \dots b_{T_{1q}\rho_{1q}}$ are signed minors of $B = (b_{T_{1\rho_1}})$. The equations (25) represent a non-singular composition on $T_{1,q+1} \dots T_{1n}$ if B is non-singular³¹ on T_1, ρ_1 . If $q = n$, we let³²

$$(26_1) \quad z_{T_{\sigma 1}}^{(1)} \rightarrow \Delta z_{T_{\sigma 1}}^{(1)},$$

$$(26_2) \quad z_{T_{\sigma 2}}^{(2)} \rightarrow z_{T_{\sigma 2}}^{(2)},$$

$$\vdots$$

$$(26_q) \quad z_{T_{\sigma q}}^{(q)} \rightarrow z_{T_{\sigma q}}^{(q)},$$

³⁰ For two-way compounds see Cullis, "Matrices and Determinoids," vol. 1, p. 289.

³¹ Cullis, *Ibid.* This property is used by J. Williamson in his paper entitled "Matrices whose sth compounds are equal," Bulletin of the American Mathematical Society, vol. 39 (1933), p. 108.

³² " $\rightarrow$ " means "go into."

where $\Delta = |b_{T_1 \rho_1}|$, and the determinants of the transformations (26₁), (26₂), $\dots$, (26_q) are $\Delta^n, 1, \dots, 1$ respectively. This completes the case where a composition is made on a signant partition of $F(A)$.

To treat the remaining case let $C = (c_{T_\sigma \rho_1})$. Let further

$$F(A | T_\sigma | C) = \sum_{\substack{(T_{11} \dots T_{1n}) \\ \vdots \\ (T_{\sigma-1,1} \dots T_{\sigma-1,n})}} [T_{11} \dots T_{1n}] \dots [T_{\sigma-1,1} \dots T_{\sigma-1,n}] a_{T_{11} \dots T_{\sigma 1}} \dots a_{T_{1q} \dots T_{\sigma q}} c_{T_{\sigma 1} \rho_{11}} \dots c_{T_{\sigma p} \rho_{1q}} y_{T_{1, q+1}}^{(q+1)} \dots y_{T_{1n}}^{(n)} \dots u_{T_{\sigma-1, n}}^{(n)} w_{\rho_{11}}^{(1)} \dots w_{\rho_{q1}}^{(q)}.$$

$F(A | T_\sigma | C)$ can be obtained from $F(A)$ by the compositions

$$\begin{aligned} z_{T_{\sigma 1}}^{(1)} &= \sum_{\rho_{11}} c_{T_{\sigma 1} \rho_{11}} w_{\rho_{11}}^{(1)}, \\ &\vdots \\ z_{T_{\sigma q}}^{(q)} &= \sum_{\rho_{1q}} c_{T_{\sigma q} \rho_{1q}} w_{\rho_{1q}}^{(q)}, \end{aligned} \quad (27)$$

where C is the matrix of composition in each of the equations of (27). If C is non-singular on T_σ, ρ_1 , the above compositions are non-singular on $T_{\sigma 1}, \rho_{11}, \dots, T_{\sigma q}, \rho_{1q}$ respectively.

The above treatment can be extended at once to the cases where $T_1, \dots, T_\sigma$ are not of the same orders, and the non-singularity of B and C on T_1, ρ_1 and T_σ, ρ_1 respectively can be replaced by non-singularity on T_1 and T_σ . We have now proved that compositions of A on T_i , or on partitions contained in T_i ($i = 1, \dots, \sigma$), with matrices non-singular on these partitions, correspond to compositions on the $(T_1 \dots T_{\sigma-1}, T_\sigma)$ -compound of A with matrices non-singular on the partitions on which these compositions are made; hence the space ranks of the $(T_1 \dots T_{\sigma-1}, T_\sigma)$ -compound of A are invariant, in the sense of Theorem 41, under non-singular compositions on A .

Nothing further can be obtained by using derivatives with more than one partition signant.

31. Invariant factors. Let $A = (a_{i_1 \dots i_n})$ be a cube matrix of order m on each index. Let $\delta = (\delta_{i_1 \dots i_n})$, where $\delta_{i_1 \dots i_n} = 1$ when $i_1 = \dots = i_n = 1, \dots, m$, and $\delta_{i_1 \dots i_n} = 0$ for all other values of the subscripts. Let $A_1 = (a_{i_1 j_1}^{(1)}), \dots, A_n = (a_{i_n j_n}^{(n)})$ be non-singular square matrices which satisfy

$$(\delta_{i_1 \dots i_n} a_{i_1 j_1}^{(1)} \dots a_{i_n j_n}^{(n)}) = (\delta_{j_1 \dots j_n}) = (\delta_{i_1 \dots i_n}). \quad (28)$$

Let simultaneous transformations on δ with $A_1, A_2, \dots, A_n$ be called a *similar transformation*. Let λ represent a parameter. By Theorems 20 and 21 the greatest common divisor of the q th-order minors of the $(i_1 \dots i_s, i_{s+1} \dots i_n)$ -derivate of $(A - \lambda \delta)$ of any signancy is equal to the greatest common divisor of

the minors of the corresponding derivate of $(B - \lambda\delta)$, where B is obtained from A under a similar transformation. The greatest common divisor of the determinants of the r th-order minors of the $(i_1 \cdots i_s, i_{s+1} \cdots i_n)$ -derivate of $(A - \lambda\delta)$ are divisible by the greatest common divisor of the determinants of the $(r - 1)$ st order minors of this derivate. Let the quotient of these greatest common divisors be called the *r*th-invariant factor of the $(i_1 \cdots i_s, i_{s+1} \cdots i_n)$ -derivate of A . We similarly define the *r*th-invariant factor of the $(i_1 \cdots i_n; \alpha \cdots \alpha)$ -derivate of A .

We have

THEOREM 52. *The invariant factors of a matrix A are unchanged under similar transformations.*

It is readily shown that except in the two-way case, the invariant factors do not form a complete invariant system under similar transformations. In another paper the author has given such a complete system of invariants.

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